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Alternating-offer Bargaining with the Global Games Information Structure

Anton Tsoy*

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Abstract

This paper studies frequent-offer limits of perfect Bayesian equilibria in the alternating-offer bilateral bargaining model with private correlated values. The correlation of values is modeled via the global games information structure: values depend on the unobserved quality of the object and idiosyncratic factors. For any level of correlation we construct a punishing path that exhibits the Coasian dynamics and enables to sustain a variety of outcomes even when the correlation of values is almost perfect. We characterize the Pareto frontier of frequent-offer PBE limits as the correlation approaches perfect and show that such limits exhibit no delay, but the surplus split generally differs from that in the complete-information game. We also construct frequent-offer PBE limits that exhibit trade delays even when the correlation of values is close to perfect. Our findings highlight the role of public information for bargaining delays.

Keywords: *bargaining delay, alternating offers, incomplete information, private correlated values, Coase conjecture, global games.*

JEL Code: *C74, C78, D82.*

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1 Introduction

In many markets, prices are negotiated bilaterally and parties' privately known valuations are correlated. Examples include over-the-counter (OTC) markets for financial assets, real estate, private equity, and durable goods such as cars, private jets, etc. In such markets, values depend on both the unobserved "quality" of the object of trade and idiosyncratic factors. For example, in OTC markets for financial assets, such as corporate bonds or collateralized debt obligations, the price a trader is willing to pay or accept for an asset depends on the risks associated with it as well as the trader's portfolio strategy and hedging needs. Differences in values can also arise due to discrepancies in the subjective evaluations of the asset by traders. The bargaining literature has so far focused exclusively on one-sided or two-sided independent private information, but has left unaddressed the case of two-sided correlated private information.

This paper studies frequent-offer limits[1] of perfect Bayesian equilibria (PBE) of the alternating-offer bilateral bargaining model with private correlated values. We show that even when values become almost perfectly correlated, the bargaining outcome can differ from the complete-information outcome in both the split of the surplus and delay. In this limit, while players' information about values is precise, a variety of bargaining outcomes is possible because of the lack of common knowledge about values. This result stresses the role of public rather than private information for predicting the bargaining outcome, and shows that the Nash bargaining solution (Nash (1953)) commonly used in applications as a reduced form for the complete-information bargaining outcome may be less compelling in opaque markets with scarce public information, e.g. OTC markets.

The correlation of values is captured via the global games information structure commonly used in the global games literature (see Morris and Shin (1998, 2001)). Specifically, players' types are uniformly distributed on a "diagonal stripe" inside the unit square (Figure 1). Values are private: players' values are strictly increasing functions of their own types. Thus, the buyer with a higher value assigns positive probability to an interval of seller's types with higher costs. In the OTC example, the buyer of the asset with a high valuation attributes it partially to low risks associated with the asset, and hence, predicts that the seller's valuation is also relatively high.[2] The global games information structure

[1] The focus on frequent-offer limits is standard in the bargaining literature for a number of reasons including the focus on sources of bargaining delay beyond the exogenously assumed infrequency of offers, robustness of such limits to details of the bargaining protocol (e.g. Rubinstein (1982), Abreu and Gul (2000)), and technical challenges in formulating games directly in continuous time (Simon and Stinchcombe (1989)).

[2] The private values assumption is relevant in OTC markets e.g. when bargaining occurs through

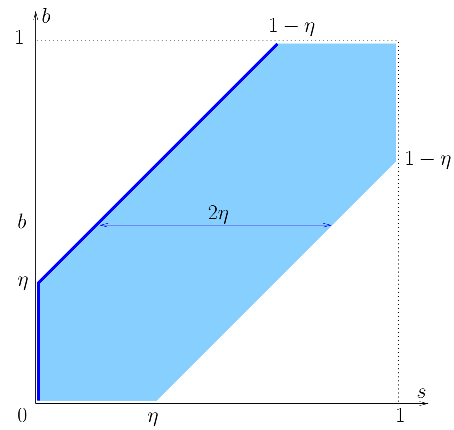


Figure 1: ***Distribution of types.*** *Types (s, b) are uniformly distributed on the diagonal stripe of width 2η inside the unit square. The bold line depicts the support of optimistic conjectures of the buyer.*

is quite rich and incorporates a variety of correlation levels ranging from almost perfect correlation (very narrow stripe) to independent values (very wide stripe).[3] We consider the case of no adverse selection, i.e. for any realization of types gains from trade are positive.[4]

Given the strong notion of the correlation of values and no adverse selection, one may expect that as the correlation of values becomes almost perfect, the bargaining outcome converges to the complete information outcome studied in Rubinstein (1982), in which equally patient players immediately split the surplus equally. Surprisingly, we show that even for almost perfect correlation, a variety of bargaining outcomes can be sustained as frequent-offer PBE limits and we obtain the type of multiplicity that is common in bargaining models with two-sided independent private information about values. To shed light on this multiplicity recall that with independent values, multiple outcomes can be sustained as frequent-offer PBE limits with *optimistic conjectures* in which, for example, after the seller's deviation, each type of the buyer puts probability one on the lowest

brokers that trade on behalf of their clients. Clients value the asset directly and the asset quality determines the price they are willing to pay or accept for the asset. The brokers' payoffs equal the difference between their clients' values and the actual price of trade. Since brokers' values are not derived directly from the asset, the private-values assumption is justified.

[3]We also allow for a rich class of mappings from types into values. This way, it is possible to match the empirical marginal distribution of values of both sides by varying the mappings from types into values. In this respect, the distributional assumption is not as restrictive as it might seem at first.

[4]This assumption is realistic in OTC markets where participants trade to manage liquidity. In asset markets, the seller can be forced to liquidate her positions because of the urgent need to raise cash, and other things equal, her value of the asset is lower than the value of the buyer who is not hit by the liquidity shock.

type of the seller. With such beliefs, there is a continuation equilibrium with the Coasian property giving the lowest (over all continuations) share of the surplus to the deviator. When values are highly correlated however such optimistic conjectures result only in a marginal belief updating (the diagonal stripe in Figure 1 is very narrow). We show that nevertheless there is a continuation equilibrium with optimistic conjectures with the following *contagious Coasian property*: in the frequent-offer limit it gives as harsh punishment to the deviator as in the case of independent values. The key observation is that the Coasian property obtains in the game between the lowest type of the seller and a narrow range of buyer types that under optimistic conjectures assign probability one to the lowest type of the seller. The contagion argument allows us then to extend the Coasian property for types at the bottom to the rest of types.

Further, we characterize the Pareto frontier of frequent-offer PBE limits as the correlation of values approaches perfect. On this Pareto frontier, the bargaining outcomes are efficient as in the complete information case, however, the ex-ante split of the surplus between equally patient players in general differs from equal. To characterize the Pareto frontier we construct PBEs with the following *segmentation dynamics*. There is a number of endogenously defined segments and players with their first offers announce segments. If players agree on the segment, they trade at the price corresponding to the segment, otherwise, the war-of-attrition type of dynamics emerges: both sides continue insisting on their initial announcements until one of the sides gives in. Initial announcements allow players to quickly establish the common knowledge of a small set of types and thus avoid the inefficient bargaining delay. Moreover, by increasing the number of segments, we can construct PBEs that approximate any point on the Pareto frontier or approximate the complete information outcome.

While bargaining outcomes on the Pareto frontier feature immediate trade, we also construct inefficient frequent-offer PBE limits that exhibit delay even when the correlation of values is close to perfect. Such outcomes exhibit realistic *two-sided screening dynamics*. Both sides start from extreme price offers and gradually converge in their offers. All types on each side pool on price offers, but separate by the time they give in and accept the opponent's offer. Unlike in PBEs with segmentation dynamics, offers do not reveal the private information and trade happens through gradual acceptance leading to the inefficient bargaining delays. In a companion paper (Tsoy (2016)), we show that when types are affiliated and the distribution of types has full support, under a weak version of the requirement that the support of beliefs cannot expand,[5] bargaining delays are necessary

[5]The assumption of non-expanding support is common in the bargaining literature (see Bikhchandani

in separating frequent-offer PBE limits with almost perfect correlation. Moreover, under this alternative model of correlation, two-sided screening dynamics can still be sustained. Therefore, we conclude that PBEs with segmentation dynamics are not robust to the model of the correlation of values, while PBEs with two-sided screening dynamics are.

The persistence and robustness of the two-sided screening dynamics sheds light on the role of public information in bargaining which has not been emphasized in the bargaining literature. In our model, when the correlation is high, the private information of players is quite precise, while the public information about values remains relatively crude (more precisely, values are not common knowledge). This aspect is important in OTC asset markets where only limited public information in the form of credit ratings, benchmarks, past quotes, etc., is available about assets. At the same time, professional traders are "sophisticated" and rely in addition on their private information sources to evaluate the asset riskiness. One might be tempted to conclude that since the information of traders is more refined when compared to the public information, the public information is extraneous in OTC markets and transparency is not necessary. Our analysis suggests that the precision of public rather than private information is important to guarantee that the complete-information bargaining game is a good approximation of such environments, and in particular, to ensure that the trade is efficient.

Related Literature Our paper is related to several strands of literature. The Nash bargaining solution is widely used in applications to determine the bargaining outcome. The axiomatic bargaining problem studied in Nash (1953) gives the prediction about the split of the surplus, and further non-cooperative analysis (Rubinstein (1982), Binmore et al. (1986)) shows that when values are common knowledge, this split can be attained without delay which is how the Nash bargaining solution is used in applications. We show that when values are *almost* common knowledge, the bargaining outcome can be quite different both in terms of bargaining delay and price of trade, and so one has to be more cautious in applications where the public information is crude, such as in OTC markets. In this respect, our paper is also related to the literature exploring the effect of uncertainty and higher-order uncertainty on bargaining outcomes. The literature on the Coase conjecture shows that the bargaining outcome is sensitive to even a small amount of private information (see e.g. Fudenberg et al. (1985), Gul and Sonnenschein (1988), Grossman and Perry (1986), Ausubel and Deneckere (1992a), Gul et al. (1986)). Feinberg and Skrzypacz (2005) shows that the Coase conjecture itself is not robust to

(1992), Grossman and Perry (1986), Rubinstein (1985a)).

the second-order uncertainty. While the previous literature assumed big differences in the support of players' beliefs to obtain the discontinuity in the equilibrium outcomes, in this paper, as correlation becomes almost perfect, the supports of players' beliefs become concentrated around the realized types. Weinstein and Yildiz (2013) shows that the complete-information game is not robust to the perturbations of higher-order beliefs, however, their result involves complex and somewhat artificial types, while our type space has a natural interpretation.

The bargaining literature with two-sided private information about values focuses exclusively on independent values. In such models, generally a variety of equilibrium dynamics can be sustained with optimistic conjectures, and the literature studied particular classes of equilibria and restricted attention to one-sided offers.[6] Ausubel and Deneckere (1992b) shows that without the gap between the lowest values of players, in a rich subclass of sequential equilibria the screening delay is degenerate and essentially no trade happens as offers become frequent.[7] Cramton (1984) constructs an equilibrium where first the seller gradually reveals her type and then screens buyer types. Cho (1990) considers a class of equilibria in which the seller's price offers perfectly separate types in every round. Both equilibria in Cramton (1984) and Cho (1990) converge to immediate trade at price equal to the lowest valuation of the buyer as offers become frequent. Cramton (1992) considers the model with two-sided offers where parties strategically choose the amount of delay to signal their values.[8] Our paper complements this literature by constructing a rich class of equilibria with segmentation and two-sided screening dynamics in the new environment with correlated values and two-sided offers and showing that a variety of bargaining outcomes is possible even with little private information.

Deneckere and Liang (2006), Fuchs and Skrzypacz (2013), Gerardi et al. (2014) explored the model with one-sided private information about interdependent values.[9] In that model, only one party knows the quality of the object which determines values of both parties. When the uninformed party makes offers, there is a unique equilibrium similarly to the Coasian-style models. Our paper differs from this literature in that both

[6]The exception is Ausubel and Deneckere (1993) which allowed offers by both sides and justified that the restriction to one-sided offers from the welfare perspective.

[7]They also construct relatively efficient equilibria in which the static monopoly outcome is realized.

[8]See also Fudenberg and Tirole (1983) for the analysis of the model with two bargaining rounds, and Chatterjee and Samuelson (1987) for a neat characterization of the bargaining dynamics under the additional restriction that the type and action space consist of only two types and two offers. Watson (1998) analyzed the uncertainty about discount factors.

[9]An earlier analysis of this model is given in Vincent (1989).

parties have private information and this information is correlated.[10]

The segmentation dynamics of PBEs approximating the Pareto frontier are similar to the war-of-attrition dynamics in reputational bargaining (Abreu and Gul (2000), Kambe (1999), Compte and Jehiel (2002), Wolitzky (2012), Fanning (2016)). In their models, commitment types require a fixed share of the surplus. This results into the war-of-attrition, in which rational types mimic certain commitment types. In our model such dynamics arise despite the fact that all types are rational. Our paper also gives the connection between the trade dynamics and primitives such as values of players which is important for applications, while the reputational bargaining literature is silent about where the commitment types come from.

Although we use the information structure from the global games literature (Carlsson and Van Damme (1993), Morris and Shin (1998, 2001)), our results about multiplicity of limits is quite different from the selection results in the global games literature.[11] Morris and Shin (2012) shows the contagious adverse selection can lead to a market break down in a static trading game. Differently, we combine the contagion argument with the Coasian argument to show that even when players have a great flexibility in exchanging offers, the inefficient trade delay may arise as an equilibrium outcome. Similarly to Morris and Shin (2012), we stress that the public information ensures efficiency through building common knowledge among players, as opposed to reducing the adverse selection which was studied e.g. in Daley and Green (2012), Duffie et al. (2014), Asriyan et al. (2015).

The paper is organized as follows. Section 2 describes the game. Section 3 derives the optimal punishment for off-path deviations. Section 4 characterizes the Pareto frontier as the correlation becomes almost perfect and shows that bargaining delay may persist even when the correlation is almost perfect. Section 5 concludes with the discussion of the the role of public information in bargaining, robustness of different equilibria, order of limits, and directions for future research. To maintain continuity of the argument, we relegate all proofs to the Appendix.

[10] The relevance of either model depends on the application. While the model with interdependent values can be applied to study negotiations in primary markets where there is an asymmetry of information between the originator of the security or asset and the buyer, our model better describes the negotiations in secondary markets where both sides have some private information arising from their liquidity needs and differences in the evaluation of the asset.

[11] This is not entirely surprising given that the contagion argument has less bite in dynamic environments (see Angeletos et al. (2007), Chassang (2010)).

2 The Model

The buyer (he) and the seller (she) negotiate the price of the indivisible good. The seller's type s and the buyer's type b are jointly uniformly distributed on the diagonal stripe inside the unit square $\Omega_\eta = \{(s,b) \in [0,1]^2 : s-\eta \leq b \leq s+\eta\}$ (see Figure 1). The uncertainty parameter $\eta \in (0,1]$ controls the degree of the correlation of types. Let $\overline{\pi}(x) = \min\{1, x+\eta\}$ and $\underline{\pi}(x) = \max\{0, x-\eta\}$. Given their types, players' prior beliefs about the opponent's type are uniform on $B_s = [\underline{\pi}(s), \overline{\pi}(s)]$ for the seller type s and on $S_b = [\underline{\pi}(b), \overline{\pi}(b)]$ for the buyer type b .

Such an information structure is similar to the global games information structure and it captures the correlation of values in many markets. For example, in OTC financial markets, one can think of the asset quality ω reflecting the risks associated with the asset and types being players' private signals $s = \omega + \eta_s$ and $b = \omega + \eta_b$ about the asset quality, where η_b and η_s reflect both discrepancies in the risk evaluation and idiosyncratic factors affecting valuations, such as portfolio strategy and hedging needs. By varying η, the model spans a variety of environments. When players have precise information about each others' values (e.g. negotiations between sophisticated bond traders that can evaluate very precisely risks associated with the bond), η could be thought of as small. In applications where idiosyncratic components are big (e.g. the acquisition of a company), η could be thought of as large.

The valuation of the good of a type b buyer is $v(b)$, and the cost of selling the good for a type s seller is $c(s)$ where $v : [0,1] \to \mathbb{R}$ and $c : [0,1] \to \mathbb{R}$ are strictly increasing, continuously differentiable functions with derivatives bounded from below and above by some positive constants.[12] Monotonicity of v and c implies that values are positively correlated. The uncertainty about the type of opponent translates into the uncertainty about the opponent's value. We assume that gains from trade are positive for any realized types. Note that this does not preclude the possibility that $c(1) > v(0)$, and hence, there does not in general exist a single price that gives non-negative utility to all types.

Bargaining occurs in rounds $n = 1, 2, \ldots$. The length of the time interval between bargaining rounds is $\Delta > 0$. Players discount the future at common discount rate $r > 0$. Denote by $\delta = e^{-r\Delta}$ the common discount factor. The seller is active in odd rounds, and

[12] When types are independent ($\eta = 1$), it is without loss of generality to assume that types are uniformly distributed. For any distribution of values, there is a transformation of functions v and c that preserves the distribution of values and changes the distribution of types into uniform on the unit interval. With correlated types this is no longer true as no such transformation is guaranteed to preserves the correlation. We consider a general class of functions v and c, but restrict the distribution of types to be uniform.

the buyer is active in even rounds. An active player can either accept the last offer of the opponent or make a counter-offer. Once a price offer is accepted, the game ends and payoffs are determined. If trade happens in round N at price p, the buyer's utility is $\delta^{N-1}(v(b)-p)$ and the seller's utility is $\delta^{N-1}(p-c(s))$.[13]

In any round n by the beginning of which trade has not happened, a history h^n is a sequence of rejected price offers up to round $n-1$. A (pure) strategy of the buyer σ_b^n is a measurable function that maps any buyer type b and history h^n into the acceptance decision or a counter-offer. The posterior beliefs of the buyer μ_b^n is a measurable function that maps any buyer type b and any history h^n into a probability distribution over seller types. The strategy σ_s^n and the posterior beliefs μ_s^n are defined analogously for the seller.[14]

A *perfect Bayesian equilibrium (PBE)* consists of a pair of strategy profiles (σ_b^n, σ_s^n) and beliefs (μ_b^n, μ_s^n) that satisfy the sequential rationality and following conditions on beliefs: (a) Bayes' rule is applied to update beliefs whenever possible; (b) μ_b^n and μ_s^n do not change in even and odd rounds, resp.; (c) for any history h^n, $\mu_b^n \in \Delta(S_b)$ and $\mu_s^n \in \Delta(B_s)$. This is a natural adaptation of the perfect Bayesian equilibrium (Fudenberg and Tirole (1991)) to the environment with correlated values. The last requirement states that both on and off the equilibrium path, players put positive probability only on types of the opponent that they initially considered possible, i.e. in B_s or S_b.

We focus on limits of equilibria as offers become frequent, i.e. $\Delta \to 0$ or equivalently $\delta \to 1$. An *outcome* in the bargaining game is a mapping from types (s, b) into the time of trade τ and the price ρ. A *PBE outcome* is the outcome induced by equilibrium strategies, i.e. $(N\Delta, p)$. We call (τ, ρ) the *frequent-offer PBE limit* of a sequence of PBEs indexed by $\delta \to 1$ if equilibrium outcomes $(N\Delta, p)$ converge in probability to the outcome (τ, ρ) as $\delta \to 1$, i.e. for any $\varepsilon > 0$, $\lim_{\delta \to 1} \mathbb{P}\left(|N\Delta - \tau| > \varepsilon \text{ or } |p - \rho| > \varepsilon\right) = 0$.

When $\eta \to 0$, the correlation of values becomes almost perfect. In this limit, the model approaches the complete-information bargaining game analyzed in Rubinstein (1982), in the sense that infinite hierarchies of beliefs of types s and b, or Harsanyi's types (Harsanyi (1967)), approach in the product topology types with common knowledge of values $c(s)$ and $v(b)$. Note however that no matter how small η is, it is only common knowledge among

[13]By convention, if trade does not occur in a finite number of rounds, $N = \infty$ and both players get a payoff of zero.

[14]It is standard in the bargaining literature to restrict attention to equilibria in pure strategies with the reservation that mixing is possible off the equilibrium path (see Gul et al. (1986), Fudenberg et al. (1985) for a discussion of mixing off-path). In this paper mixing could be necessary only for seller type 0 and buyer type 1 out of the equilibrium path of the continuation equilibrium with optimistic conjectures analyzed in the next section. With additional notations our results can be formulated to incorporate this possibility.

players that types are in Ω_η and in this sense the public information remains coarse even as the uncertainty about values vanishes. We say that the outcome (τ, ρ) is the *almost-public information limit* if there is a sequence of frequent-offer PBE limits (τ_η, ρ_η) indexed by $\eta \to 0$ such that for any $\varepsilon > 0$, $\lim_{\eta\to 0} \mathbb{P}_\eta\left(|\tau_\eta - \tau| > \varepsilon \text{ or } |\rho_\eta - \rho| > \varepsilon\right) = 0$ where $\mathbb{P}_\eta$ is the uniform distribution on Ω_η. We use the complete-information model as a benchmark to compare with the almost-public information limits of our model. In the complete-information model, the PBE is unique and converges to the outcome $(0, \frac{1}{2}(v(b) + c(s))$ as $\delta \to 1$ (Rubinstein (1982)). The complete-information outcome provides non-cooperative foundations to the axiomatic Nash bargaining solution (Nash (1953)) and we refer to the equal split of the surplus as the *Nash split*.

3 Contagious Coasian Property

This section derives the optimal punishment for detectable deviations.[15] Even when $\eta \approx 0$ and players assign positive probability only to a very narrow range of opponent's values, the punishment is as harsh as in the case of large uncertainty ($\eta \approx 1$). The approach is to first derive weak lower bounds on equilibrium payoffs and then show that the continuation equilibrium with optimistic conjectures attains these lower bounds as offers become frequent.

Let $\overline{y}(s,b) = \frac{\delta c(s)+v(b)}{1+\delta}$ and $\underline{y}(s,b) = \frac{c(s)+\delta v(b)}{1+\delta}$. These are the equilibrium offers of the seller and the buyer, respectively, when values $v(b)$ and $c(s)$ are common knowledge (Rubinstein (1982)). In the complete information game, the first offer is accepted, and as $\delta \to 1$, $\overline{y}(s,b)$ and $\underline{y}(s,b)$ converge to an equal split of the surplus which we denote by $y^*(s,b) = {}^1\!/_2(c(s) + v(b))$. The next lemma gives weak bounds on equilibrium prices.

Lemma 1. *In any PBE and after any history,*

(a) the seller accepts with probability one any offer above $\underline{y}(1,1)$*, and the buyer rejects with probability one any offer above* $\overline{y}(1,1)$*;*

(b) the buyer accepts with probability one any offer below $\overline{y}(0,0)$*, and the seller rejects with probability one any offer below* $\underline{y}(0,0)$*;*

(c) the seller's continuation utility is at least $\max\{\underline{y}(0,0) - c(s), 0\}$ *and the buyer's continuation utility is at least* $\max\{v(b) - \overline{y}(1,1), 0\}$*.*

[15] An action is a *detectable deviation* if all types of the opponent detect it, i.e. assign to it probability zero on the equilibrium path.

When there is a big variation in values across types, $\underline{y}(0,0)$ and $\overline{y}(1,1)$ are far apart and Lemma 1 puts only weak bounds on the price of trade even when η is very small. These bounds are standard in the bargaining literature (e.g. Grossman and Perry (1986), Watson (1998)). By Lemma 1, the seller cannot do better than if she convinces the buyer that her cost is the highest possible and the buyer admits that his value is the highest possible. Moreover, the buyer always has the option to trade immediately at price $\underline{y}(1,1)$ by admitting that he has the highest valuation and by recognizing that the seller has the highest cost.

When offers are frequent, Lemma 1 implies that the seller trades only at prices above $y^*(0,0)$. Moreover, the seller can secure utility 0 by rejecting any offer and making unacceptable counter-offers above $\overline{y}(1,1)$. Hence, the seller's utility is at least $\max\{y^*(0,0) - c(s), 0\}$ in any continuation equilibrium. Symmetrically, the buyer's utility is at least $\max\{v(b) - y^*(1,1), 0\}$. One might expect that when η is small and values are *almost* common knowledge, players can secure a higher level of utility. Indeed, Rubinstein (1982) shows that when values *are* common knowledge and offers are frequent, both players are guaranteed $1/2(v(b) - c(s))$ in any continuation equilibrium. The next theorem shows that this is not the case. (We formulate the theorem for the seller and symmetric result holds for the buyer).

Theorem 1 (**Contagious Coasian Property**). *Consider a history h that contains a detectable deviation of the seller, but not the buyer and after which posterior beliefs of the seller are truncated from above at some $\overline{b}$. For any $\varepsilon > 0$, there exists $\overline{\delta}$ such that for all $\delta > \overline{\delta}$, there exists a continuation equilibrium in which the continuation utility of any seller type s is at most* $\max\{y^*(0,0) - c(s), 0\} + \varepsilon$.

Theorem 1 derives an optimal punishment for detectable deviations with several useful properties. First, the punishment is the harshest possible. It attains the lower bound on the seller's utility of the seller given by Lemma 1. Second, a single equilibrium punishes all types of the seller simultaneously, hence, the buyer does not need to detect which type of the seller deviated. Finally, the convergence is uniform in types ($\overline{\delta}$ does not depend on s).

In the proof of Theorem 1, we construct the continuation equilibrium with *optimistic conjectures* of the buyer. Specifically, the seller has her original beliefs, while the buyer puts probability one on the lowest type in the support S_b, i.e.

$$\mu_b^n[\underline{\pi}(b)] = 1, \tag{1}$$

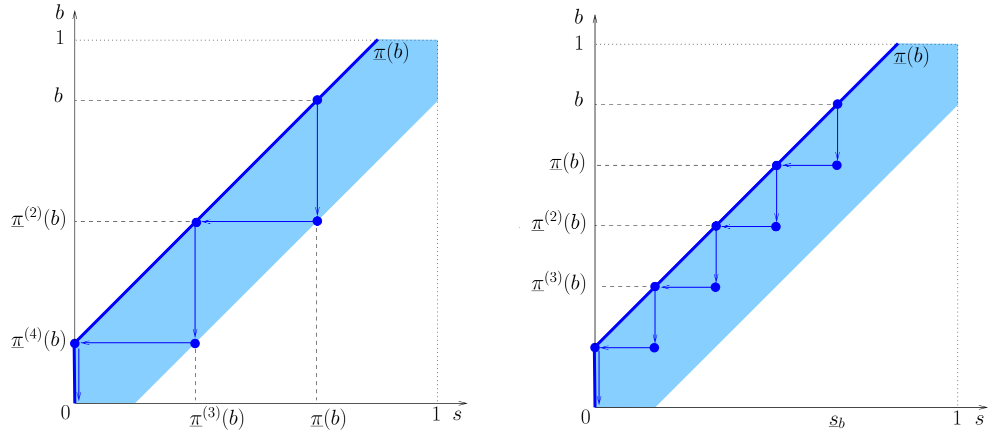
for all $b \in [0,1]$ and all histories with a detectable deviation of the seller.[16] Beliefs (1) are a natural counterpart of optimistic conjectures commonly used in the bargaining literature (e.g. Rubinstein (1985b)).

When $\eta = 1$, optimistic conjectures lead to a very drastic updating of beliefs: all types of the buyer put probability one on type 0 of the seller. Hence, the model is reduced to the one-sided incomplete information game between the informed buyer with a type $b \in [0,1]$ and the seller with known cost $c(0)$ analyzed in Grossman and Perry (1986), Gul and Sonnenschein (1988). In this case, Theorem 1 is the standard Coasian property stating that as $\delta \to 1$, the screening by type 0 happens in a short period of time and all screening offers are close to $y^*(0,0)$. Hence, the seller gets utility close to $\max\{y^*(0,0) - c(s), 0\}$. The essence of the Coasian argument is that when the seller cannot commit to future offers, even the monopolistic seller faces the competition from her own future offers. As offers become frequent and hence the competition from future offers more severe, the seller looses the power to price discriminate and allocates almost immediately at the lowest price.

When $\eta < 1$, only types of the buyer below η put probability one on type 0 of the seller and so, the standard Coasian property implies willingness to pay close to $y^*(0,0)$ only for these types. When η is small, this is only a small interval of types, and it is not a priori obvious whether optimistic conjectures provide as harsh of a punishment as when $\eta = 1$. The surprising conclusion of Theorem 1 is that the same punishment is possible for any η, and in particular for arbitrary small η, providing offers are sufficiently frequent. For general η, the proof combines the contagion argument from the global games literature with the Coasian argument from the bargaining literature. The proof is quite involved and here we simply give the intuition for how the contagion allows us to leverage the Coasian forces.

We construct the continuation equilibrium under optimistic conjectures in which all types of the buyer pool on the lowest acceptable price $\underline{y}(0,0)$ and accept offers according to the *willingness to pay* $P(b)$ which is the price at which type b is indifferent between accepting and rejecting. The seller optimally screens the buyer with the screening policy

[16] Such beliefs can be justified by the following trembles in the model with a finite number of types and finite grid of price offers. Seller's and buyer's types come from $\{k/K\}_{k=1}^{K}$ for some integer K. Suppose price offers come from a discrete set $\mathcal{P}$. Seller type s trembles with probability $(1-s)^m/2$ for some integer m and conditional on trembling, she chooses a price offer uniformly from $\mathcal{P}$. As $m \to \infty$, the probability of tremble converges to zero. Yet, conditional on the buyer type b, the probability that the tremble comes from seller type $\underline{\pi}(b)$ is $\frac{(1-\underline{\pi}(b))^n}{(1-\underline{\pi}(b))^m + \sum_{s \in S_b \setminus \{\underline{\pi}(b)\}} (1-s)^m} \to 1$ as $m \to \infty$, since $\frac{1-s}{1-\underline{\pi}(b)} < 1$.



(a) *No screening:* $t(s) = \underline{\pi}(s)$ (b) *Selling above s:* $t(s) = s$

Figure 2: ***Contagion mechanism.*** *Bold line* $\underline{\pi}(b)$ *depicts the support of buyer's optimistic conjectures. Horizontal arrows represent beliefs of the buyer, vertical arrows represent the first screening cutoff of the seller. E.g. buyer type* b *puts probability one on type* $\underline{\pi}(b)$ *of the seller and expects her to allocate to all types above* $t(\underline{\pi}(b))$ *in the first round of screening.*

that depends on her type.[17] We denote by $t(s)$ the lowest buyer type who buys in the first round of screening by type s of the seller and so, in the first round of screening type s of the seller makes the price offer $P(t(s))$ and all buyer types above $t(s)$ accept it. We consider the continuation equilibrium in which functions $P(\cdot)$ and $t(\cdot)$ are right-continuous and strictly increasing. The main step of the proof is to show that the willingness to pay $P(\cdot)$ converges uniformly over types of the buyer to $\max\{y^*(0,0), c(\underline{\pi}(b))\}$ as $\delta \to 1$ which implies Theorem 1.

First, consider how the willingness to pay function $P(\cdot)$ is determined. All buyer types below η put probability one on type 0 of the seller and for them $P(\cdot)$ is pinned down by the screening policy of type 0. For types above η, if type b rejects $P(b)$, then he is still the highest type in the support of beliefs of type $\underline{\pi}(b)$, as only types above b accept $P(b)$. Hence, after the rejection of $P(b)$, seller type $\underline{\pi}(b)$ will restart screening and again offer $P(t(\underline{\pi}(b)))$ which gives

$$P(b) = (1 - \delta^2)v(b) + \delta^2 P(t(\underline{\pi}(b))), \tag{2}$$

for $b \in (\eta, 1]$.

Now, we can illustrate the contagion mechanism. Suppose first that the seller does

[17] The Online Appendix provides the existence result.

not screen, i.e. $t(s) = \underline{\pi}(s)$, and sells to all types in B_s immediately by offering $P(\underline{\pi}(s))$ in the first round of screening (see Figure 2a). Using $t(s) = \underline{\pi}(s)$, we can extrapolate (2) to get for $b > \eta$

$$\begin{aligned} P(b) &= (1-\delta^2)v(b) + \delta^2 P(\underline{\pi}^{(2)}(b)) \\ &= (1-\delta^2)v(b) + (1-\delta^2)\delta^2 v(\underline{\pi}^{(2)}(b)) + \delta^4 P(\underline{\pi}^{(4)}(b)) \\ &= (1-\delta^2)\sum_{k=0}^{K-1} \delta^{2k} v(\underline{\pi}^{(2k)}(b)) + \delta^{2K} P(0), \end{aligned} \tag{3}$$

where $\underline{\pi}^{(k)}$ is the k-th iteration of function $\underline{\pi}$ and K is the smallest k such that $\underline{\pi}^{(2k)}(b) = 0$. Formula (3) captures the essence of the contagion mechanism. Buyer type b puts probability one on the seller type $\underline{\pi}(b)$ and expects to pay $P(t(\underline{\pi}(b))) = P(\underline{\pi}^{(2)}(b))$ in the first round of screening. In turn, buyer type $\underline{\pi}^{(2)}(b)$ puts probability one on the seller type $\underline{\pi}^{(3)}(b)$ and expects to pay $P(t(\underline{\pi}^{(3)}(b))) = P(\underline{\pi}^{(4)}(b))$ in the first round of screening, and so on, until eventually the seller type 0 is reached. When η is small, K can be large, and hence, for a fixed δ the first term in equation (3) can be significant. However, it vanishes as $\delta \to 1$ and $P(b) \to P(0)$ for all b. This way if the Coasian property holds at the bottom, i.e. $P(b) \approx y^*(0,0)$ for $b \in [0,\eta]$ as $\delta \to 1$, then the low willingness to pay of types in $[0,\eta]$ translates into the low willingness to pay of all the rest buyer types.

Of course, in equilibrium, the seller can optimally choose more discriminatory screening policy. In fact, for a given willingness to pay function $P(\cdot)$, as $\delta \to 1$, the seller becomes more patient and hence, *will* screen more finely. In turn, more fine screening by the seller leads to higher $P(b)$. To see this effect, consider a more discriminatory screening policy $t(s) = s$ (see Figure 2b). Again extrapolating (2), for $b > \eta$

$$\begin{aligned} P(b) &= (1-\delta^2)v(b) + \delta^2 P(\underline{\pi}(b)) \\ &= (1-\delta^2)v(b) + (1-\delta^2)\delta^2 v(\underline{\pi}(b)) + \delta^4 P(\underline{\pi}^{(2)}(b)) \\ &= (1-\delta^2)\sum_{k=0}^{K'-1} \delta^{2k} v(\underline{\pi}^{(k)}(b)) + \delta^{2K'} P(0), \end{aligned} \tag{4}$$

where K' is the smallest k such that $\underline{\pi}^{(k)}(b) = 0$. It is easy to see that for every b, expression (4) for $P(b)$ is higher than (3). Hence, a finer screening policy increases the willingness to pay of the buyer.

Therefore, there are two opposing forces that affect $P(\cdot)$. On the one hand, because of the contagion mechanism the willingness to pay decreases as $\delta \to 1$. On the other hand, the seller screens more finely as $\delta \to 1$ which increases the willingness to pay. Theorem 1 shows that the former force dominates using the following Coasian argument. As in the case $\eta = 1$, we can analyze the game between types of buyer in $[0, \eta]$ and type 0 of the seller, to which they assign probability one, as a separate bargaining game with one-sided incomplete information. For such game, the standard Coasian property implies that $P(b) \to y^*(0,0)$ as $\delta \to 1$. Now, for type s_1 of the seller slightly above 0, a big fraction of the types of the buyer that she faces belongs to $[0, \eta]$ and has willingness to pay close to $y^*(0,0)$. The profit from types below η creates a sufficient temptation for the seller type s_1 to quickly screen types above η. But this implies that types of the buyer slightly above η have the option to buy at price close to $y^*(0,0)$ after a short delay, which in turn implies that the willingness to pay of types in $[0, s_1 + \eta]$ is also close to $y^*(0,0)$. Similarly, we can show that type s_2 of the seller slightly above s_1 quickly screens all types above $s_1 + \eta$ and so, the willingness to pay of types in $[0, s_2 + \eta]$ is also close to $y^*(0,0)$. Proceeding in this fashion, we can show that the willingness to pay of all types is the lowest possible in the limit as $\delta \to 1$.[18] To summarize, the Coasian property at the bottom extends through the contagion mechanism to all types.

4 Almost-Public Information Limits

Theorem 1 shows that the optimal punishment in the limit of frequent offers does not depend on the level of correlation. This suggests that a variety of on-path equilibrium dynamics is possible as offers become frequent even when the correlation of values is close to perfect. This section studies bargaining outcomes in the almost-public information limit, and for this limit, answers the questions: 1) is trade always efficient and if not, is efficient trade possible; 2) do efficient outcomes coincide with the complete-information outcome and if not, can they approximate the complete-information outcome.

[18] The argument is more subtle than described. If type b of the buyer puts probability one on the seller type $\underline{\pi}(b)$ with costs above $y^*(0,0)$, then allocating at price close to $y^*(0,0)$ is no longer a temptation for the seller type $\underline{\pi}(b)$. Hence, such type will always make offers above $c(\underline{\pi}(b))$ and can potentially screen indefinitely. For readers familiar with the bargaining literature, this case resembles the "no-gap" case, while the case described above is the "gap" case. We show in the proof of Theorem 1 that despite the fact that in this case, the willingness to pay is higher than $y^*(0,0)$ in the limit, it increases just enough to cover the costs $c(\underline{\pi}(b))$.

4.1 Pareto Frontier

In this subsection, we characterize the Pareto frontier of allocations sustainable as almost-public information limits. Although such limits can approximate the complete-information outcome, in general there is a variety of other efficient surplus splits that can be attained by the almost-public information limits.

Let $\Pi = \mathbb{E}_{(s,b)}[v(b) - c(s)]$ be the expected surplus. For any bargaining outcome (τ, ρ), let $U^S = \mathbb{E}[e^{-r\tau}(\rho - c(s))]$ and $U^B = \mathbb{E}[e^{-r\tau}(v(b) - \rho)]$ be players' expected utilities at the ex-ante stage (before types are realized). By Lemma 1, in the limit $\delta \to 1$, the seller's expected utility cannot be lower than $\underline{U}^S = \mathbb{E}_{(s,b)}[\max\{y^*(0,0), c(s)\} - c(s)]$ or exceed $\overline{U}^S = \mathbb{E}_{(s,b)}[\min\{y^*(1,1), v(b)\} - c(s)]$. Thus, the set of efficient allocations that are potentially sustainable as almost-public information limits is a subset of $PF = \{(U^S, U^B) : U^S + U^B = \Pi \text{ and } U^S \in [\underline{U}^S, \overline{U}^S]\}$. The next theorem shows that the Pareto frontier of almost-public information limits coincides with PF.

Theorem 2 (Pareto Frontier). *1. Consider $(U^S, U^B) \in PF$. There exists an almost-public information limit with ex-ante expected utilities of players (U^S, U^B).*

2. The complete-information outcome $(0, y^*(s, b))$ is an almost-public information limit.

The first part of Theorem 2 shows that even when we restrict attention to efficient outcomes, the bargaining outcome is sensitive to the amount of public information. When values are common knowledge, equally patient players split the surplus equally without delay, i.e. $U^S = U^B = \frac{1}{2}\Pi$ (Rubinstein (1982)). In contrast, when values are almost common knowledge a range of ex-ante surplus splits is sustainable as frequent-offer PBE limits. This range is determined by how close the bounds on prices $y^*(0,0)$ and $y^*(1,1)$ are to each other. When $y^*(0,0) \approx y^*(1,1)$, the outcome is close to the equal split, as in this case it is common knowledge that values are close. However, when $y^*(0,0)$ and $y^*(1,1)$ are far apart, the ex-ante split of surplus can be very far from equal, even when values are almost common knowledge and players face little uncertainty about each others values ($\eta \approx 0$). The second part of Theorem 2 states that it is still possible to approximate the complete information outcome $(0, y^*(s, b))$ with frequent-offer PBE limits when the correlation is close to perfect. In other words, for $\eta \approx 0$ we can construct PBEs, in which for any realization of types the surplus split is close to equal and the trade delay is short.

The proof of Theorems 2 applies the construction of frequent-offer PBE limits in which players with their initial offers establish common knowledge of a relatively narrow range of types and this way avoid the inefficient trade delay. In the remainder of the section, we outline the construction of such PBE limits.

Consider first the following auxiliary continuous-time war-of-attrition game $\mathcal{G}$. The buyer and the seller make offers q^B and q^S, resp., irrespective of type. The game stops once one of the parties accepts the opponent's offer and trade happens at the accepted price. We denote such game by $\mathcal{G}(q^S, q^B)$. Threshold acceptance strategies are described by a strictly decreasing b_t and strictly increasing s_t such that for any t, all buyer types above b_t and all seller types below s_t accept the opponent's offer. This game has the following Bayesian Nash equilibrium (BNE) in threshold strategies.

Lemma 2. *Suppose* $s_\infty \in (\eta, 1-\eta)$,$b_\infty = s_\infty - \eta$,$\max\{c(s_\infty), y^*(0,0)\} < q^B < v(0)$, *and* $c(1) < q^S < \min\{v(b_\infty), y^*(1,1)\}$. *Consider thresholds strategies* b_t *and* s_t

$$r(v(b_t) - q^S) = \tfrac{\dot{s}_t}{b_t - s_t + \eta}(q^S - q^B), \tag{5}$$

$$r(q^B - c(s_t)) = -\tfrac{\dot{b}_t}{b_t - s_t + \eta}(q^S - q^B), \tag{6}$$

with the terminal conditions $\lim_{t\to\infty} s_t = s_\infty$ *and* $\lim_{t\to\infty} b_t = b_\infty$. *Then threshold strategies* b_t *and* s_t *constitute BNE in the war-of-attrition game* $\mathcal{G}(q^S, q^B)$ *and* $b_0 \leq s_0 + \eta$. *Moreover, the ex-ante probability of delay is at most* 6η.

We interpret the BNE in Lemma 2 as follows. There are two segments: one with a higher price q^S and one with a lower price q^B. Each side insists on belonging to the segment with a more favorable price by delaying the trade and the war-of-attrition dynamics emerges. Each side gradually accepts the opponent's offer starting from the top for the buyer and from the bottom for the seller. At every time t, threshold types are indifferent between accepting the current opponent's offer and marginally delaying the acceptance. In equations (5) and (6), the costs of marginal delay due to discounting (left-hand side) equal for threshold types the benefits of marginal delay due to the opponent giving in first (right-hand side). Since types are positively correlated, the likelihood of acceptance depends on the type: a higher type of the seller assigns lower probability to the buyer's acceptance. The fact that more impatient players also believe that the opponent is tougher guarantees that threshold strategies are indeed optimal. Further, it implies that types of the buyer above $s_\infty + 2\eta$ and types of the seller below $b_\infty - 2\eta$ assign probability zero to their offer being accepted and they should accept immediately. This guarantees that the probability of delay is of order of magnitude η and so, the outcome is close to efficient when $\eta \approx 0$.

Our war-of-attrition bargaining game resembles the war-of-attrition dynamics in the reputational bargaining . Unlike the reputational bargaining, where demands are given by exogenous demands of commitment types, in our model all types are rational. Hence, in our model, there is no upper bound on the time of trade (otherwise a rational player would prefer to wait slightly past the final round of bargaining), while in the standard model of reputational bargaining (Abreu and Gul (2000)), rational types accept by some final time.

Denote by $\nu^p(s_\infty, q^S, q^B)$ the bargaining outcome determined by the BNE described in Lemma 2. The next lemma shows that this outcome can be approximated by the PBE outcomes in our discrete-time bargaining game as $\delta \to 1$.

Lemma 3. *Suppose* $s_\infty \in (\eta, 1-\eta)$,$b_\infty = s_\infty - \eta$,$\max\{c(s_\infty), y^*(0,0)\} < q^B < v(0)$, *and* $c(1) < q^S < \min\{v(b_\infty), y^*(1,1)\}$. *There is a frequent-offer PBE limit which coincides with* $\nu^p(s_\infty, q^S, q^B)$.

In the proof of Lemma 3 we construct threshold strategies in discrete time that approximate the continuous time threshold strategies,[19] and then use the contagious Coasian property of the punishing paths to sustain the on-path strategies. (Restrictions on q^S and q^B in Lemma 3 guarantee that on-path continuation utilities exceeds the lower bounds on players' utilities given in Lemma 1 at any moment in time).

Of course, the type of outcome described in Lemma 2 may not always be possible (e.g. when $c(1) > v(0)$). Thus, the next step is to repeat the construction in Lemma 3 and increase the number of segments. On the one hand, this guarantees that the trade is efficient in the limit as the correlation becomes almost perfect. On the other hand, this allows us to choose at what prices different types trade and hence, match the Nash split or the division of the ex-ante surplus. The following lemma generalizes the construction in Lemma 3.

Lemma 4. *Fix an integer* Z*, a strictly increasing sequence of offers* $\{q_z\}_{z=1}^Z \subset (y^*(0,0), y^*(1,1))$, *and increasing sequences* $\{b^z\}_{z=1}^Z$ *and* $\{s^z\}_{z=1}^Z$ *such that 1)* $b^0 = s^0 = 0$, $b^Z = s^Z = 1$; *2)* $s^z = b^z + \eta$ *and* $c(s^z) < q_z < v(b^{z-1})$ *for* $z = 1, \ldots, Z$; *3)* $b^z - b^{z-1} > 4\eta$ *for* $z = 1, \ldots, Z$. *There exists a frequent-offer PBE limit in which the ex-ante probability of delay is at most* $6\eta(Z-1)$, *and types of the buyer in* $[b^z, b^{z+1}]$ *and types of the seller in* $[s^z, s^{z+1}]$ *trade only at price* q_z *or* q_{z+1}.

[19] This step is quite involved technically, as one needs to show that a system of difference equations admits a monotone solution. The assumption that the distribution of types is uniform on Ω_η (as opposed to general affiliated) is used only in the proof of this result.

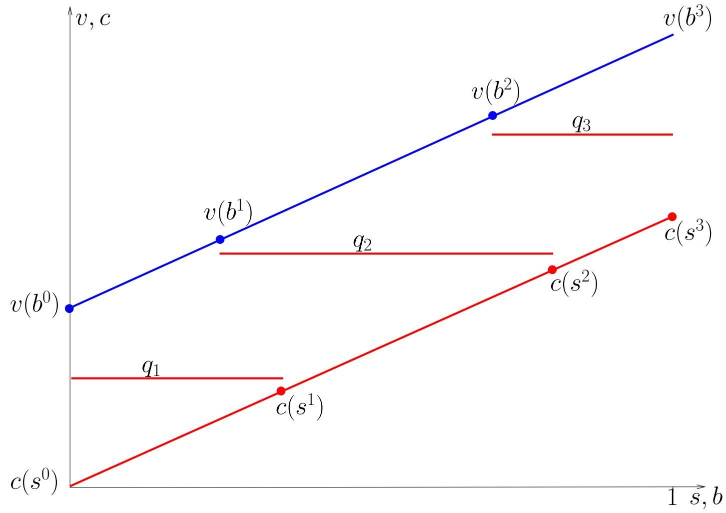


Figure 3: ***Example of offers and boundary types*** s^z ***and*** b^z ***satisfying requirements of Lemma*** 4.

Figure 3 illustrates the offers and threshold types that satisfy the conditions of Lemma 4, which are generally can be easily satisfied for some sequences of b^z, s^z and q_z when η is sufficiently small. To prove Lemma 4, we add to the war-of-attrition game described above the initial stage at which players can announce one of Z segments where each segment is identified with a price $q_z, z = 1, \ldots, Z$. If players agree on the segment, then trade happens at the price of this segment. If they disagree and announce adjacent segments, say q_z and q_{z+1}, then the war-of-attrition bargaining game $\mathcal{G}(q_{z+1}, q_z)$ starts with offers fixed at the initial announcements. If they disagree and announced non-adjacent segments, then each side has an option to accept the initial offer, but otherwise, announcements happen again immediately. We choose segments sufficiently far apart ($b^{z+1} - b^z > 4\eta$) so that each type knows that the opponent can announce one of at most two neighboring segments, which guarantees that deviations to non-adjacent segments are out of equilibrium path. In the proof of Lemma 4, we construct a discrete-time analogue of such a war-of-attrition with initial announcement and show that it constitutes the PBE for sufficiently large δ.

By Lemma 3, the probability of the disagreement for any particular pair of prices q_z and q_{z+1} is at most 6η. Because the inefficient delay occurs only when the announcements differ, the total probability of delay is at most $6\eta(Z-1)$. Therefore, when Z is small relative to $1/\eta$ (e.g. when Z is constant), the outcome is again efficient in the limit $\eta \to 0$.

Notice that here the assumption that the support of beliefs is bounded plays a crucial role in the segmentation dynamics. For example, when the buyer observes an unexpectedly high segment announcement, he simply ignores it. With the full-support of beliefs this would not be the case: if such an announcement is on-path, then the buyer should update his beliefs that the seller's costs are (unexpectedly) high. We return to this issue when we discuss the plausibility of difference bargaining outcomes in Section 5.

Finally, we can apply Lemma 4 to prove Theorem 2. Let

$$q(\iota, \gamma, \beta) = (1-\gamma) y^*(\iota, \iota) + \gamma\beta \max\{c(\iota), y(0,0)\} + \gamma(1-\beta)\min\{v(\iota), y^*(1,1)\}. \quad (7)$$

For any γ and β, we can apply Lemma 4 to construct frequent-offer PBE limits with segments that are $\sqrt{\eta}$ apart (hence $Z \sim \frac{1}{\sqrt{\eta}}$) and prices $q_z = p(s^z, \gamma, \beta)$. As $\eta \to 0$, the probability of delay is bounded from above by $6\eta(Z-1) \sim \sqrt{\eta}$ and converges to zero. The length of each segment $\sqrt{\eta}$ also converges to zero and so each type of the seller s trades at a price close to $q(s, \gamma, \beta)$. Since $q(\iota, 0, \beta) = y^*(\iota, \iota)$ and $[\underline{U}^S, \overline{U}^S] = \{\mathbb{E}_{(s,b)}[q(s,1,\beta) - c(s)], \beta \in [0,1]\}$, with appropriate choices of γ and β we can get the desired approximations in Theorem 2.

4.2 Inefficient Delay

This subsection shows that the frequent-offer PBE limits need not converge to efficiency no matter how close the correlation is to perfect. Specifically, we construct almost-public information limits that approximate the Nash split, but do so with the delay.

Let us revisit the continuous-time war-of-attrition game $\mathcal{G}$ introduced in the previous section, but suppose now that price offers instead of being fixed are given by a strictly decreasing function q_t^S for the seller and a strictly decreasing function q_t^B for the buyer such that $q_T^S = q_T^B$ for some $T < \infty$. As before, players choose the time when they accept the opponent's offer. Denote such a game by $\mathcal{G}(q_t^S, q_t^B)$. We again focus on BNEs in threshold acceptance strategies.

Lemma 5. *Suppose there exist thresholds strategies b_t and s_t that satisfy*

$$r(v(b_t) - q_t^S) = -\dot{q}_t^S, \quad (8)$$

$$r(q_t^B - c(s_t)) = \dot{q}_t^B; \quad (9)$$

with the terminal condition $b_T = s_T + \eta$. Then b_t and s_t constitute a BNE in the war-of-attrition game $\mathcal{G}(q_t^S, q_t^B)$.

The incentives for delay in the BNE described in Lemma 5 are the opposite of that in the war-of-attrition game with constant offers studied in the previous subsection. When offers are constant, the benefits from the delay come from the acceptance of the opponent. Thus, only types that assign a positive probability to each other's acceptance delay the trade, and as the interval of such types is at most 2η, the outcome approaches efficiency when $\eta \to 0$. In contrast, in the BNE described in Lemma 5, before the final time T (when price offers are equal and bargaining stops), each side assigns probability zero to its offer being accepted and the only incentive for delaying the trade comes from waiting for a more favorable opponent's offer. Thus, the two-sided screening dynamics emerges. For types above b_T, the buyer is screened by a path of decreasing seller's offers q_t^S, while for types below s_T, the seller is screened by a path of increasing buyer's offers q_t^B. Here, the optimality of threshold strategies follows from the standard skimming property: higher types of the buyer and lower types of the seller are more impatient and thus, accept earlier. Notice that because of the two-sided screening dynamics it is necessary that there is a delay, as long as $q_0^B < q_T^B$ and $q_0^S > q_T^S$, and as a result, the outcome is inefficient.

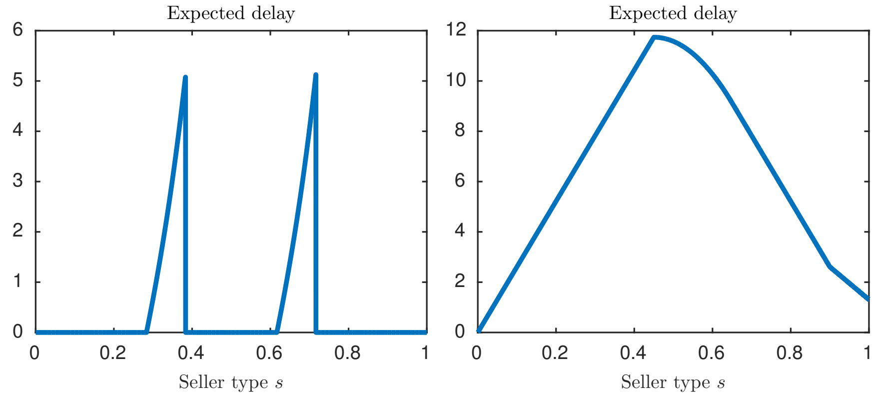
We can choose price offer paths in such a way that for small η, the prices of trade approximate the Nash split. Specifically, we set $q_t^S = \frac{1}{2}(v(b_t - \frac{\eta}{2}) + c(b_t - \frac{\eta}{2}))$, $q_t^B = \frac{1}{2}(v(s_t + \frac{\eta}{2}) + c(s_t + \frac{\eta}{2}))$, and b_t and s_t that solve the system (8) $-$ (9) with the initial condition $b_0 = 1$ and terminal condition $b_T = s_T + \eta$. Denote such an outcome by $(\tau_\eta^\dagger, \rho_\eta^\dagger)$. The next theorem shows that such an outcome can be approximated by the frequent-offer PBE limits.

Theorem 3 (**Inefficient limit**)**.** *There is a frequent-offer PBE limit which coincides with* $(\tau_\eta^\dagger, \rho_\eta^\dagger)$*. Moreover,* $\lim_{\eta\to 0} \mathbb{P}_\eta\left(|\rho_\eta^\dagger - y^*(s,b)| > \varepsilon\right) = 0$ *for all* $\varepsilon > 0$*, and* $\lim_{\eta\to 0} \mathbb{E}_\eta[\tau_\eta^\dagger] > \underline{\tau}$ *for some* $\underline{\tau} > 0$*.*

Theorem 3 shows that there is an inefficient almost-public information limit that approximates the Nash split of the surplus. Together with Theorem 2, this result shows that the complete-information outcome is not a good approximation for environments where players have little uncertainty about each other's values ($\eta \approx 0$), as long as the public information is crude. This is true even if we additionally restrict attention to efficient outcomes or outcomes consistent with the axiomatic Nash bargaining solution.

We already have all the ingredients to prove Theorem 3. We construct equilibria in grim trigger strategies. Players start from the main path and stick to it as long as there are no detectable deviations. The main path is the discretization of continuous-time paths b_t, s_t, q_t^B, q_t^S. If a detectable deviation happens, then the deviator is punished by the continuation equilibrium with optimistic conjectures described in Theorem 1.

There is one detail about the construction of the main path worth stressing. Rather



(a) Segmentation dynamics (b) Screening dynamics

Figure 4: ***Expected delay in segmentation and screening limits.***

than having each side make offers that are gradually accepted by opponent (as in the war-of-attrition game $\mathcal{G}(q_t^S, q_t^B)$), we construct on-path PBE strategies in which both sides make non-serious offers until one of the sides reveals an interval of types by making an acceptable offer. The construction is such that as $\delta \to 1$, at any time t only threshold types b_t and s_t make "serious" offers q_t^S and q_t^B, resp., while all the rest types pool on some un-acceptable offer. The reason for this signaling rather than screening on-path dynamics is as follows. If say the seller were to make screening offers, then after some histories in which the buyer deviated from his acceptance strategy, the seller would have to make offers that bring her negative payoff. This would contradict the sequential rationality after such off-path histories. This problem does not arise with the signaling on-path dynamics. If say the buyer deviates from her revelation strategy, then in equilibrium he corrects his deviation in the next round. Thus, the seller is never forced to make an offer that brings her negative payoff when accepted.

Comparison of the segmentation and screening dynamics As an illustration of the segmentation and screening dynamics, consider the following example. Let $v(b) = b+\frac{2}{3}$, $c(s) = s$,$\eta = 0.1$, and $r = 10\%$. Figure 4a depicts the expected delay in the frequent-offer PBE limit described in Lemma 4 with three segments: $q^1 = \frac{1}{2}$,$q^2 = \frac{5}{6}$,$q^3 = \frac{3}{2}$ and $b^1 = \frac{1}{3} - \frac{\eta}{2}$, $s^1 = \frac{1}{3} + \frac{\eta}{2}$,$b^2 = \frac{2}{3} - \frac{\eta}{2}$, $s^2 = \frac{2}{3} + \frac{\eta}{2}$. Figure 4$b$ depicts the expected delay in the frequent-offer PBE limit described in Theorem 3 with price paths $q_t^S = \frac{5}{3} - (\frac{1}{3} + \frac{\eta}{2})(1 + tr)$ and $q_t^B = (\frac{1}{3} + \frac{\eta}{2})(1 + tr)$ and corresponding acceptance strategies solving the system $(8) - (9)$ with $b_T = \frac{1}{2} + \frac{\eta}{2}$ and $b_T = s_T + \eta$ as terminal conditions. The expected delay is non-monotone in the player's type and it is highest for types close to the boundaries of segments in the segmentation dynamics and for types close to the middle of the range of types in the screening dynamics. To measure the efficiency, we use the share of surplus that is dissipated due to trade delay $\frac{\mathbb{E}_{(s,b)}[e^{-r\tau}(v(b)-c(s))]}{\mathbb{E}_{(s,b)}[v(b)-c(s)]}$. In the segmentation limit, the trade delay occurs only for types close to boundaries of segments and only 3.1% of surplus is dissipated. In the screening limit, the trade delay occurs for almost all types due to gradual convergence of prices, and 45.5% of the surplus is dissipated due to bargaining delay. We see that even for relatively low players' uncertainty about values, the bargaining outcomes can differ substantially depending on the type of equilibrium played and need converge to the complete-information outcome.

5 Discussion

This paper studies the alternating-offer bargaining model with the global games information structure. Despite the strong notion of correlation – each type assigns positive probability only to a set of types of opponent – a variety of equilibrium dynamics can be sustained even when the correlation of values is close to perfect. In the almost-public information limit, the outcome is neither guaranteed to be efficient nor on the Pareto frontier, the split of the surplus is guaranteed to coincide with the complete-information split. To conclude, we discuss the implications of our analysis, the robustness of screening and segmentation dynamics, and the order of limits.

The Role of Public Information for Bargaining Efficiency The non-robustness of the complete-information bargaining model highlights the role of public information in bargaining. In the almost-public information limit, the players' uncertainty about values is vanishingly small, but the public information about values is crude (e.g. it is only common knowledge that the buyer's value is in $[v(0), v(1)]$). Many secondary markets, such as OTC markets for financial assets, are known for their opaqueness and lack of transparency. It is sometimes argued that because traders in such markets are sophisticated, they are capable of overcoming this lack of public information about assets. In fact, the growing literature on search-and-bargaining models of OTC markets (e.g. Duffie et al. (2005), Lagos and Rocheteau (2009)) applies the Nash bargaining solution to capture the bargaining outcome, thus, implicitly assuming that the complete-information model is a good approximation for OTC negotiations. Our analysis shows that the lack of common knowledge may force both sides to start the negotiation from extreme offers and enter into an inefficient phase of exchange of offers. Hence, the sophistication of traders in OTC markets need not compensate for the market opaqueness. In a companion paper (Tsoy (2016)), we show that bargaining delays arising from the lack of public information lead to a very different predictions about the market liquidity compared to the standard search-and-bargaining models of OTC markets.

Robustness to the Model of Correlation The companion paper (Tsoy (2016)) considers an alternative model of correlation in which the types are distributed according to the affiliated distribution with full support on the unit square. We additionally use a slightly weaker version of the standard refinement in the bargaining literature (see Bikhchandani (1992), Grossman and Perry (1986), Rubinstein (1985a)) that the support of beliefs does not expand.[20] The main result is that in the almost-public information limit, the delay is necessary to attain the Nash split. This shows the non-robustness of the efficient bargaining outcomes constructed in Theorem 2. In fact, the source of this non-robustness is clear from our construction of PBEs approximating the Pareto frontier. There, it is important that every side expects at most two offers from the opponent and offers that corresponds to distant segments are ignored. This allows players to quickly coordinate on a narrow band of types and results in an efficient outcome. With the full support of beliefs this is not possible anymore, as offers corresponding to distant segments would be on-path. Further, we show in Tsoy (2016)) that the two-sided screening dynamics in Theorem 3 can be still sustained in the model with the full support of beliefs. We conclude that the almost-public information limits with two-sided screening dynamics are more robust to the assumptions about the details of the correlation of values, than the efficient segmentation dynamics.

Order of Limits This paper demonstrates lack of upper-hemicontinuity in the frequent-offer PBE limits: for arbitrary small η, there is a variety of outcomes possible, however, when $\eta = 0$, there is a unique frequent-offer PBE limit. We focus in the paper on the order of limits where we first take limit $\delta \to 1$ and then take limit $\eta \to 0$. The next theorem shows that under the reversed order of limit, all PBEs converge to the outcome of the complete-information game as one would expect.

[20] We do not impose this refinement in the present paper, as the bounded support of beliefs already restricts significantly the evolution of beliefs after deviations. However, the full-support beliefs have an undesirable property that the initial correlation of values is virtually erased out of the equilibrium path. Even if beliefs of each type are initially concentrated on a very narrow range of types of opponent, after the deviation, beliefs need not preserve the initial correlation of values. Hence, the refinement is desirable in such a model.

Theorem 4. *For any δ, consider a sequence of PBEs indexed by $\eta \to 0$ that converges to the limit $(\tau_\delta, \rho_\delta)$. Then for any $\varepsilon > 0$, $\lim_{\delta \to 1} \mathbb{P}_{\eta=1}(\tau_\delta > \varepsilon$ and $|\rho_\delta - y^*(s,b)| > \varepsilon) = 0$.*

In our view, the order of limits used in this paper is more relevant in applications. The complete-information game corresponding to $\eta = 0$ is well understood: the first offer is $\overline{y}(s,b)$ and it is accepted by the buyer. At the same time, little is known about the bargaining with correlated private information. A natural way to explore it is by looking at the limit when offers are frequent, and hence, the delay comes purely from the private information of parties.

There are several interesting directions for future research. First, the prediction that the amount of public information can affect bargaining delays for a fixed precision of the private information can be tested in the lab.[21] Second, we prove the contagious Coasian property for any η, but focus on the analysis of the equilibrium set in the limit $\eta \to 0$, as the correlation of values is the novel feature of our environment. However, our construction of frequent-offer PBE limits with segmentation and screening dynamics is valid for a range of ηs. The analysis of the equilibrium set for intermediary levels of correlation is an interesting avenue for future research. Third, another theoretical development of the model is to explore the predictions of the model in the presence of outside options, as in Fuchs and Skrzypacz (2010), or to endogenize the length of bargaining rounds and use an intuitive criterion style refinement, as in Admati and Perry (1987) and Cramton (1992). Finally, the delay generated in model with almost-public information about quality proves very tractable in applications, as in our model one can essentially abstract from private information and still generate non-trivial delays. Tsoy (2016) uses it to study the implications of bargaining delay for liquidity in OTC financial markets, and it is interesting to study implications of bargaining delays in different applications, such as labor or monetary economics.

A Appendix

This Appendix contains proofs of the main results. Auxiliary technical lemmas and additional results are provided in the Online Appendix.

[21] In an early study, Roth and Murnighan (1982) study experimentally how the fact that the information is common knowledge or not affects bargaining outcomes.

A.1 Proofs for Section 3

We first derive general bounds on acceptable offers that imply Lemma 1. Let P_b be the supremum of offers accepted by type b of the buyer with positive probability in equilibrium (both on- and off-path), and analogously, let P_s be the supremum of offers rejected by type s of the seller with positive probability in equilibrium, p_b be the infimum of prices rejected by type b of the buyer with positive probability in equilibrium, p_s be the infimum of prices accepted by type s of the seller with positive probability in equilibrium.

Lemma 6. *For all b, s,*

$$P_b \leq (1-\delta)\sum_{k=0}^{\infty}\delta^{2k}\left(v(\overline{\pi}^{(2k)}(b)) + \delta c(\overline{\pi}^{(2k+1)}(b))\right),$$
$$p_s \geq (1-\delta)\sum_{k=0}^{\infty}\delta^{2k}\left(c(\underline{\pi}^{(2k)}(s)) + \delta v(\underline{\pi}^{(2k+1)}(s))\right).$$

Proof. First, by the definition of P_s, type b of buyer can guarantee himself the utility arbitrarily close to $\delta(v(b) - \max_{s\in S_b} P_s)$ by making an offer arbitrarily close to $\max_{s\in S_b} P_s$ whenever he is active. Hence, $\delta(v(b) - \max_{s\in S_b} P_s) \leq v(b) - P_b$. Second, let U_s be the supremum of the continuation utilities of type s on- and off-path if the trade does not occur in the current round. If type s of the seller rejects an offer, she cannot guarantee more than $\max\{\delta(\max_{b\in B_s} P_b - c(s)), \delta^2 U_s\}$, which implies $U_s \leq \delta(\max_{b\in B_s} P_b - c(s))$. Hence, $P_s - c(s) \leq \delta(\max_{b\in B_s} P_b - c(s))$. Therefore,

$$\begin{aligned} P_b &\leq (1-\delta)v(b) + \delta \max_{s\in S_b} P_s \\ &\leq (1-\delta)v(b) + \delta \max_{s\in S_b}\left((1-\delta)c(s) + \delta \max_{b'\in B_s} P_{b'}\right) \\ &= (1-\delta)\left(v(b) + \delta c(\overline{\pi}(b))\right) + \delta^2 \max_{s\in S_b}\max_{b'\in B_s} P_{b'}. \end{aligned}$$

By iterating this inequality, we obtain the first inequality in the statement of the lemma. The argument for the second inequality is symmetric. □

Proof of Lemma 1. Follows from Lemma 6 and the monotonicity of functions v and c. □

Now, we analyze the *punishing equilibrium*, the continuation equilibrium with optimistic conjectures of the buyer that we construct to prove Theorem 1. Below we first describe strategies in the punishing equilibrium and then prove the contagious Coasian property of such a continuation equilibrium. The existence of the punishing equilibrium for sufficiently large δ is proven in Theorem 5 in the Online Appendix. The argument is provided for the case $\overline{b} = 1$ and can be immediately extended to the general case $\overline{b} < 1$ by simply ignoring types of the buyer above $\overline{b}$.

A.1.1 Description of Strategies

Since the optimistic conjectures of the buyer may exclude the realized seller's type, the buyer and the seller may have different expectations regarding the path of play in the punishing equilibrium. For concreteness, we refer to the path of play expected by the seller as the *equilibrium path of the punishing equilibrium.*

Buyer on-path strategy: All types of the buyer pool on $\underline{y}(0,0)$, the lowest acceptable price offer. Type b accepts any offer less than or equal to his *willingness to pay* $P(b)$ which is right-continuous and strictly increasing in b. Since $P(\cdot)$ is strictly increasing, for any history h^n without buyer's deviations, there exists a buyer type $\beta \in [0,1]$ such that only types of the buyer below β remain in the game in round n. Whenever $\beta \geq \underline{\pi}(s)$, posterior beliefs of seller type s are uniform on $B_s \cap [0,\beta]$.

Seller on-path strategy: The seller makes offers to screen the buyer's willingness to pay. Given the highest remaining type of the buyer β, the seller type $s > 0$ chooses a cut-off buyer type $t(\beta,s)$ and allocates to all remaining buyer types above $t(\beta,s)$ by making an offer $A(\beta,s) = P(t(\beta,s))$. The strategy of type 0 of the seller differs from the rest of the types, as a positive mass of buyer types in $[0,\eta]$ puts probability one on type 0. Seller type 0 accepts the buyer's offer $\underline{y}(0,0)$, whenever the highest remaining type of the buyer is below some $\overline{\beta} \in (0,\eta]$. Given the highest remaining buyer type $\beta \in (\overline{\beta},\eta]$, seller type 0 allocates to buyer types above $t(\beta,0)$ by offering $A(\beta,0) = P(t(\beta,0))$.

Strategies off-path: Deviations by the seller are ignored. Consider histories in which type b of the buyer rejects an offer below $P(b)$. If such a deviation is not detected, then type b of the buyer continues making the offer $\underline{y}(0,0)$, and accepts any price offer less than or equal to $(1-\delta^2)v(b) + \delta^2 P(t(\beta,\underline{\pi}(b)))$. If $b > \beta + 2\eta$, then type $\underline{\pi}(b)$ detects such deviation and strategies of both players are described in Lemma 7 below. If the buyer makes a price offer different from $\underline{y}(0,0)$ or $\beta < \underline{\pi}(s)$ (the deviation from the strategy $P(\cdot)$ is detected), type s of the seller switches to optimistic conjectures and assigns probability one to the highest buyer type in the support of her prior belief, i.e.

$$\mu_s^n[\overline{\pi}(s)] = 1 \tag{10}$$

for all histories h^n with detectable deviations by both sides. Lemma 7 describes equilibrium strategies with optimistic conjectures of both players.

Lemma 7. *Suppose that for some $b' \in [0,1]$, beliefs of types of the buyer above b' and all types of the seller above $\underline{\pi}(b')$ are described by (1) and (10). Then the following strategies are the equilibrium strategies for such types. After any history, type b of the buyer above b' accepts offers less than or equal to $P^B(b)$, and otherwise, makes the counter-offer $A^B(b)$. After any history type s of the seller above $\underline{\pi}(b')$ accepts offers greater than or equal to $P^S(s)$, and otherwise,* *makes the counter-offer $A^S(s)$. Functions P^B, P^S, A^B, A^S are given by*

$$P^B(b) = \begin{cases}(1-\delta)v(b)+\delta P^S(0) \\ \overline{y}(b-\eta,b)\end{cases} \qquad A^B(b) = \begin{cases}P^S(0), & \textit{for } b\in[0,\eta), \\ \underline{y}(b-\eta,b), & \textit{for } b\in[\eta,1];\end{cases}$$

$$P^S(s) = \begin{cases}\underline{y}(s,s+\eta) \\ (1-\delta)c(s)+\delta P^B(1)\end{cases} \qquad A^S(s) = \begin{cases}\overline{y}(s+\eta,s), & \textit{for } s\in[0,1-\eta], \\ P^B(1), & \textit{for } s\in(1-\eta,1].\end{cases}$$

Proof. Consider type b of the buyer above $\max\{b',\eta\}$. Type b puts probability one on type $\underline{\pi}(b)$ of the seller by (1), while seller type $\underline{\pi}(b)$ puts probability one on type b by (10). By Rubinstein (1982) strategies of these two types given in Lemma 7 constitute the subgame perfect equilibrium of the complete information game with valuation $v(b)$ and cost $c(\underline{\pi}(b))$.

Now consider type s of the seller above $1-\eta$ that put probability one on type 1 of the buyer. Type 1 of the buyer, in turn, puts probability one on type $1-\eta$ of the seller and is willing to pay $P^B(1)$. Since $P^B(1) > c(1)$, it is optimal for types of the seller above $1-\eta$ to make offer $P^B(1)$. Moreover, they are willing to pay up to $\check{P}^S(s)$ given by $\check{P}^S(s) - c(s) = \delta(P^B(1) - c(s))$. The argument for types of the buyer below η is symmetric. □

A.1.2 Proof of the Contagious Coasian Property

Before moving on to the proof of the Contagious Coasian Property, we state the optimality conditions for the willingness to pay and screening policy. The problem of seller type s can be formulated recursively. Let $\tilde{R}(\beta,s)$ for $\beta\in[\underline{\pi}(s),1]$ be the expected profit of type $s>1$ of the seller given that the highest remaining type of the buyer is β, and denote $R(\beta,s) = \tilde{R}(\beta,s)(\beta-\underline{\pi}(s))$. Then $R(\beta,s)$ is the bounded function that satisfies the Bellman equation[22,23]

$$R(\beta,s) = \max_{b\in B_s\cap[0,\beta]} \left\{(\beta-b)(P(b)-c(s)) + \delta^2 R(b,s)\right\}. \tag{11}$$

Denote by $T(\beta,s)$ the set of maximizers in (11). Say that a seller strategy $t(\beta,s)$ is a *best-reply* to the willingness to pay $P(\cdot)$, if $t(\beta,s) = \sup T(\beta,s)$ for all s and $\beta\geq\underline{\pi}(s)$. A special role in the analysis is played by the first screening cut-off and price offer which I denote by $t(s) = t(\overline{\pi}(s),s)$ and $A(s) = A(\overline{\pi}(s),s)$, resp.

The willingness to pay $P(\cdot)$ for $b\in(\eta,1]$ satisfies (2) in the main text. The willingness to pay of types of the buyer in the interval $[0,\eta]$ differs from the rest of the types, as they all assign probability one to type 0 of the seller. Both on- and off-path, it is determined by some strictly increasing and right-continuous function $P^0(\cdot)$ (see Lemma 13 in the Online Appendix).

[22] The value function is defined only on when $\overline{\pi}(s)\leq\beta$ (otherwise, type s detects the deviation and switches to the optimistic conjectures (10)).

[23] Notice that the seller discounts by δ^2, as the buyer makes non-acceptable offers.

Consider a sequence of punishing equilibria indexed by $\delta \to 1$. In the notation, we omit the dependence of quantities in the punishing equilibrium on δ. The proof of Theorem 1 builds on a number of auxiliary lemmas. Let $\Sigma = \max_{(s,b)\in\Omega}\{v(b) - c(s)\}$ $\xi = \min_{(s,b)\in\Omega}\{v(b) - c(s)\}$ be maximal and minimal, resp., gains from trade possible in the game. Let ℓ be the upper bound on the derivatives of v and c.

Lemma 8. *Suppose there is $0 < \phi < 2\eta$ and $\hat{b} \in [\eta, 1]$ such that $c(\underline{\pi}(\hat{b})) + 2\ell\phi < P(\hat{b} - \phi)$. Then $P(\hat{b} + \phi) - P(\hat{b}) < f(\phi, \delta)$, where function f does not depend on $\hat{b}$ and $\lim_{\delta\to 1} f(\phi, \delta) = 0$ for any ϕ.*

Proof. Let $\hat{s} = \underline{\pi}(\hat{b}) + \phi$. By $0 < \phi < 2\eta$, $\underline{\pi}(\hat{s}) < \hat{b} - \phi < \hat{b} + \phi = \overline{b}_{\hat{s}}$. By the Lipschitz continuity of c,

$$c(\hat{s}) + \ell\phi < c(\underline{\pi}(\hat{b})) + 2\ell\phi < P(\hat{b} - \phi). \tag{12}$$

Let $K \leq \infty$ be the first round of screening when type $\hat{s}$ of the seller makes an offer below $P(\hat{b})$. Then

$$R(\hat{b} + \phi, \hat{s}) \leq \int_{\hat{b}}^{\hat{b}+\phi} (P(b) - c(\hat{s}))db + \delta^{2K} R(\hat{b}, \hat{s}). \tag{13}$$

Fix an integer M (to be determined later) and consider an alternative screening policy in which type $\hat{s}$ makes a sequence of offers $(A_m)_{m=1}^M$ such that $A_m = v(\hat{b} + \phi) + \frac{m}{M}(c(\hat{s}) - v(\hat{b} + \phi))$ and sells with probability one in M rounds. The loss in profit from each sale is at most $\frac{\Sigma}{M}$. By the optimality of the seller's screening policy,

$$R(\hat{b}+\phi, \hat{s}) \geq \delta^{2M} \left(\int_{\underline{b}}^{\hat{b}+\phi} (P(b) - c(\hat{s}))db - \tfrac{\Sigma}{M}(\hat{b} + \phi - \underline{b}) \right) \geq \delta^{2M} \left(\int_{\underline{b}}^{\hat{b}+\phi} (P(b) - c(\hat{s}))db - \tfrac{\Sigma}{M} \right), \tag{14}$$

where $\underline{b} = \inf\{b : P(b) > c(\hat{s})\}$. Combining (13) and (14) and rearranging terms,

$$\delta^{2M} \left(\int_{\underline{b}}^{\hat{b}} (P(b) - c(\hat{s}))db - \tfrac{\Sigma}{M} \right) \leq \left(1 - \delta^{2M}\right) \int_{\hat{b}}^{\hat{b}+\phi} (P(b) - c(\hat{s}))db + \delta^{2K} R(\hat{b}, \hat{s}). \tag{15}$$

Combining (15) with the facts that $R(\hat{b}, \hat{s}) \leq \int_{\underline{b}}^{\hat{b}} (P(b) - c(\hat{s}))db$ and the surplus is bounded by Σ,

$$\delta^{2M} \left(R(\hat{b}, \hat{s}) - \tfrac{\Sigma}{M}\right) \leq \left(1 - \delta^{2M}\right) \int_{\hat{b}}^{\hat{b}+\phi} (P(b) - c(\hat{s}))db + \delta^{2K} R(\hat{b}, \hat{s}) \leq \left(1 - \delta^{2M}\right) \Sigma\phi + \delta^{2K} R(\hat{b}, \hat{s}). \tag{16}$$

When the highest remaining type is $\hat{b}$, type $\hat{s}$ of the seller can make offer $P(\hat{b} - \phi)$ which is accepted at least by types in $(\hat{b} - \phi, \hat{b})$ and so, using (12),

$$R(\hat{b}, \hat{s}) \geq \phi(P(\hat{b} - \phi) - c(\hat{s})) > \phi^2\ell > 0. \tag{17}$$

Dividing (16) by $R(\hat{b},\hat{s})$ and using (17),

$$\delta^{2K} \geq \delta^{2M} - \frac{\Sigma}{R(\hat{b},\hat{s})}\left(\frac{\delta^{2M}}{M} + \phi\left(1-\delta^{2M}\right)\right) > \delta^{2M} - \frac{\Sigma}{\phi^2 \ell}\left(\frac{\delta^{2M}}{M} + \phi\left(1-\delta^{2M}\right)\right).$$

For each δ, we choose an integer $M(\delta)$ such that $\lim_{\delta\to 1} M(\delta) = \infty$ and $\lim_{\delta \to 1} \delta^{2M(\delta)} = 1$. Type $\hat{b}+\phi$ of the buyer prefers to purchase in the first round of screening by type $\hat{s}$ rather than wait until the screening round K when price drops below $P(\hat{b})$ and so, $v(\hat{b}+\phi) - P(\hat{b}+\phi) \geq \delta^{2K}(v(\hat{b}+\phi) - P(\hat{b}))$, which implies that

$$P(\hat{b}+\phi) - P(\hat{b}) \leq (1-\delta^{2K})(v(\hat{b}+\phi) - P(\hat{b})) \leq$$
$$(1-\delta^{2K})\Sigma \leq \left(1 - \delta^{2M(\delta)} + \frac{\Sigma}{\phi^2 \ell}\left(\frac{\delta^{2M(\delta)}}{M(\delta)} + \phi\left(1-\delta^{2M(\delta)}\right)\right)\right)\Sigma. \quad (18)$$

Denoting the last expression by $f(\phi,\delta)$ gives the desired bound. □

Definition A.1. *For any $\varepsilon > 0$, a monotone function f on $[0,1]$ is* ε-continuous *if for any open interval $I \subset [f(0), f(1)]$ of length at least ε we have $f([0,1]) \cap I \neq \emptyset$.*

Lemma 9. *For any $\varepsilon > 0$, there is $\bar{\delta} \in (0,1)$ such that for all $\delta > \bar{\delta}$: (a) function $P(\cdot)$ is ε-continuous; (b) for any type s of the seller and any type $b \in B_s$ of the buyer,*

$$P(b) - P(t(b,s)) \leq \varepsilon. \quad (19)$$

Proof. Part (a): Suppose to contradiction that there exist $\varepsilon > 0, \underline{p}, \overline{p} > \underline{p} + \varepsilon$ such that for δ arbitrarily close to 1, either $P(b) \geq \overline{p}$ or $P(b) \leq \underline{p}$ for all b. By equation (2), for any b,

$$P(b) - P(t(\underline{\pi}(b))) = (1-\delta^2)(v(b) - P(t(\underline{\pi}(b))) \leq (1-\delta^2)\Sigma < \varepsilon \quad (20)$$

for δ sufficiently close to one. Let $\tilde{b} = \sup\{b : P(b) < \underline{p}\}$. Consider type $\hat{b} = \tilde{b} + \frac{c(\eta,\delta)}{2}$ and $\check{b} = \tilde{b} - \frac{c(\eta,\delta)}{2}$ where $c(\eta,\delta)$ is as in Lemma 17 in the Online Appendix. Lemma 17 implies $\check{b} > t(\underline{\pi}(\hat{b}))$ and so, $P(\hat{b}) - P(t(\underline{\pi}(\hat{b}))) > P(\hat{b}) - P(\check{b}) > \varepsilon$, which contradicts (20).

Part (b): Consider $b \in B_s$. Since $s \geq \underline{\pi}(b)$, by Lemma 15 in the Online Appendix $t(b,s) \geq t(b, \underline{\pi}(b))$. Hence,

$$P(b) - P(t(b,s)) \leq P(b) - P(t(b,\underline{\pi}(b))) = P(b) - P(t(\underline{\pi}(b))), \quad (21)$$

which by (20) is less than $\varepsilon > 0$ for sufficiently large δ. □

Lemma 10. *There is a function $h(\cdot)$ such that $\lim_{\phi\to 0} h(\phi) = 0$ and for any $\phi > 0$ there is $\bar{\delta} \in (0,1)$ such that for $\delta > \bar{\delta}$, $P(b-\phi) \leq c(\underline{\pi}(b)) + 2\ell\phi$ implies $P(b) < c(\underline{\pi}(b)) + h(\phi)$.*

Proof. The statement of lemma can be reformulated as: for any h, there exist $\phi(h)$ and $\overline{\delta} \in (0,1)$ such that for all $\phi < \phi(h)$ and $\delta > \overline{\delta}$, $P(b-\phi) \leq c(\underline{\pi}(b)) + 2\ell\phi$ implies $P(b) < c(\underline{\pi}(b)) + h(\phi)$. Suppose to contradiction there is $h > 0$ such that for any ϕ and δ, there is $\hat{b}$ such that $P(\hat{b}-\phi) \leq c(\underline{\pi}(\hat{b})) + 2\ell\phi$ and $P(\hat{b}) \geq c(\underline{\pi}(\hat{b})) + h$. Denote $\hat{s} = \underline{\pi}(\hat{b})$ and $H = P(\hat{b}) - c(\hat{s}) \geq h > 0$. Let $K \leq \infty$ be the first round of screening in which type $\hat{s}$ of the seller makes an offer below $c(\hat{s}) + 0.7H$ and this way allocates to all types of the buyer above some $\check{b}$. By Lemma 9, $P(\check{b}) > c(\hat{s}) + 0.6H$ for sufficiently large δ. Without loss of generality, suppose $2\ell\phi < 0.1H$. This implies that the mass of types in $B_{\hat{s}}$ with $P(b) > c(\hat{s}) + 0.1H$ is at most ϕ.

Lower bound on x_K: In the first K rounds of screening, type $\hat{s}$ allocates to the mass $x_K = \hat{b} - \check{b} < \phi$ of types of the buyer. Since type $\hat{b}$ prefers to buy at price $P(\hat{b})$ rather than wait until price drops to $P(\check{b})$,

$$\delta^{2K} \leq \frac{v(\hat{b}) - P(\hat{b})}{v(\hat{b}) - P(\check{b})} \leq \frac{v(\hat{b}) - c(\hat{s}) - H}{v(\hat{b}) - c(\hat{s}) - 0.7H} \leq \frac{\Sigma - H}{\Sigma - 0.7H} < 1. \tag{22}$$

Consider an alternative screening strategy, in which type $\hat{s}$ speeds up screening in the first K/M_K rounds for some positive M_K such that K/M_K is an integer. Let A_k be the price offer that type $\hat{s}$ makes in round k in the equilibrium screening strategy. Define $q_k = P(\hat{b}) + \frac{M_K k}{K}\left(A_{K-1} - P(\hat{b})\right)$, $k = 1, 2, .., K/M_K$. In the alternative strategy, type $\hat{s}$ makes offer $p_k = \min\{q_k, A_k\}$ in rounds $k \leq K/M_K$, makes offer A_K in round $K/M_K + 1$ and continues following the equilibrium strategy from then on. The total loss from using the alternative strategy is at most $0.4H\frac{M_K}{K}x_K$. Indeed, in each round the loss of type $\hat{s}$ compared to the maximum surplus that can be extracted is at most $\frac{P(\hat{b})-P(\check{b})}{K/M_K} \leq 0.4H\frac{M_K}{K}$ and she allocates to a mass x_K of the types of the buyer. Moreover, there is no loss due to discounting, as the allocation to all buyer types happens sooner under the alternative strategy than under the equilibrium strategy. At the same time, by speeding up the screening, type $\hat{s}$ gains at least $\left(\delta^{2K/M_K} - \delta^{2K}\right)V_K$, where V_K is the continuation utility of type $\hat{s}$ after she makes price offer A_K and follows the equilibrium strategy further. By the optimality of the screening strategy of type $\hat{s}$,

$$0.4H\tfrac{M_K}{K}x_K \geq \left(\delta^{2K/M_K} - \delta^{2K}\right)V_K. \tag{23}$$

Lower bound on x_L: Consider type $\check{s} = \underline{\pi}(\check{b})$ of the seller, and let $L + K$ be the first round of screening in which type $\check{s}$ makes an offer below $c(\hat{s}) + 0.4H$ and this way allocates to all types above $\tilde{b}$. By Lemma 9, $P(\tilde{b}) > c(\hat{s}) + 0.3H$ for sufficiently large δ. By the analogous argument as with K and type $\hat{s}$ above,

$$\delta^{2L} \leq \frac{v(\check{b}) - P(\check{b})}{v(\check{b}) - P(\tilde{b})} \leq \frac{v(\check{b}) - c(\hat{s}) - .6H}{v(\check{b}) - c(\hat{s}) - .4H} \leq \frac{\Sigma - 0.6H}{\Sigma - 0.4H} < 1, \tag{24}$$

and for the optimality of strategy of seller type $\check{s}$ it is necessary that

$$0.4H\frac{M_L}{L}x_L \geq \left(\delta^{2L/M_L} - \delta^{2L}\right)V_L. \tag{25}$$

In inequality (25), $x_L = \check{b} - \tilde{b}$ denotes the mass of buyer types to whom type $\check{s}$ allocates in rounds $K, \dots, K+L$, and V_L denotes the continuation utility of seller type $\check{s}$ after price offer in round L and follows the equilibrium strategy further.

Lower bound on V_K : Observe that type $\hat{s}$ can offer $c(\hat{s}) + 0.3H$ in round $K+1$ of the screening instead of following the equilibrium screening policy. The mass of buyer types that accept such price is at least x_L and so,

$$V_K \geq 0.3Hx_L. \tag{26}$$

Lower bound on V_L : Suppose that the seller allocated in previous rounds to all types of the buyer with $P(b) > c(\hat{s}) + 0.3H$ and offers $c(\hat{s}) + 0.1H$ in the current round. Denote by b'' and b' the highest and lowest types, respectively, that accept $c(\hat{s}) + 0.1H$. By Lemma 9, $P(b'') - P(b') > 0.1H$. The next claim gives the following lower bound

$$V_L \geq 0.2H\gamma(\delta). \tag{27}$$

Claim 1. *There exist a function $\gamma(\delta)$ such that $\lim_{\delta\to 1}\frac{\gamma(\delta)}{(1-\delta)^2} \in (0,\infty)$ and $b'' - b' \geq \gamma(\delta)$ for all sufficiently large δ.*

Proof. Define $\tilde{L}$ and $t_l, l = 0, \dots, \tilde{L}+1$ recursively as follows. Let $t_0 = b''$ and $t_l = t(\underline{\pi}(t_{l-1}))$ for $l = 1, \dots, \tilde{L}+1$ where $\tilde{L}$ is the largest integer such that $t_{\tilde{L}} \geq b'$. By (2),

$$P(b'') = (1-\delta^2)\sum_{l=0}^{\tilde{L}} \delta^{2l} v(t_l) + \delta^{2(\tilde{L}+1)} P(t_{\tilde{L}+1}).$$

Since P is strictly increasing and $b' \in [t_{\tilde{L}+1}, t_{\tilde{L}}]$,

$$P(b'') - P(b') \leq (1-\delta^2)\sum_{l=0}^{\tilde{L}} \delta^{2l} v(t_l) - (1-\delta^{2(\tilde{L}+1)})P(b') \leq (1-\delta^{2(\tilde{L}+1)})(v(b'') - P(b')).$$

Since $v(b'') - P(b') \geq P(b'') - P(b') > 0,1H > 0$, $1 - \delta^{2(\tilde{L}+1)}$ is bounded away from zero. Hence, there exists $C_1 > 0$ and δ_1 such that $\tilde{L} \geq -C_1/\ln\delta$ for all $\delta \geq \delta_1$. By Lemma 17 in the Online Appendix, there exists $C_2 > 0$ and δ_2 such that $t_{l-1} - t_l > C_2(1-\delta)^3$ for all $l \in 1, \dots, \tilde{L}$ and all $\delta \geq \delta_2$. Hence, $b'' - b' = \sum_{l=1}^{\tilde{L}}(t_{l-1} - t_l) + t_{\tilde{L}} - b' \geq C_2(1-\delta)^3\tilde{L} \geq -C_1C_2(1-\delta)^3/\ln\delta \sim (1-\delta)^2$ for $\delta \geq \max\{\delta_1, \delta_2\}$. Function $\gamma(\delta) = -C_1C_2(1-\delta)^3/\ln\delta$ satisfies the desired properties. *Q.E.D.*

Multiplying inequalities (23), (25), (26), (27),

$$\tfrac{8}{3}x_K \geq \frac{K(1-\delta)}{M_K}\frac{L(1-\delta)}{M_L}\left(\delta^{2K/M_K}-\delta^{2K}\right)\left(\delta^{2L/M_L}-\delta^{2L}\right)\frac{\gamma(\delta)}{(1-\delta)^2}. \tag{28}$$

Notice that K depends on δ and $\hat{b}$, but we can choose $M_K(\delta,\hat{b})$ so that $\lim_{\delta\to 1}\delta^{K/M_K(\delta,\hat{b})}$ is a positive number in $(0,1)$. Similarly, L depends on δ and $\check{b}$, but we can choose $M_L(\delta,\check{b})$ so that $\lim_{\delta\to 1}\delta^{L/M_L(\delta,\check{b})}$ is a positive number in $(0,1)$. This way, by $H > h > 0$ and bounds (22) and (24), the right-hand side of (28) converges to a positive number as $\delta \to 1$. On the other hand, the left-hand of (28) is bounded from above by $\frac{8}{3}\phi$. Since ϕ can be arbitrarily small, we get the contradiction. □

Lemma 11. $\lim_{\delta\to 1}\max_{b\in[0,1]}|P(b)-P^*(b)| = 0$ *where* $P^*(b) = \max\{y^*(0,0), c(\underline{\pi}(b))\}$.

Proof. Fix $\varepsilon > 0$. We show that there exists $\overline{\delta}$ such that for all $\delta > \overline{\delta}$, either $P(b) < c(\underline{\pi}(b)) + \varepsilon$ or $P(b) < y^*(0,0) + \varepsilon$. Choose ϕ so that $\phi < \frac{\xi}{4\ell}$ and $h(\phi)$ defined in Lemma 10 is less than $\frac{\varepsilon}{2}$. Choose δ large enough so that $f(\phi,\delta)/\phi < \frac{\varepsilon}{2}$ where f is the function from Lemma 8.

Suppose b is such that $P(b) \geq c(\underline{\pi}(b)) + \varepsilon$. Consider the first positive integer K such that $P(b-K\phi) \leq c(\underline{\pi}(b-(K-1)\phi))+2\ell\phi$. We show that in fact there is no such an integer. By Lemma 10, $P(b-\phi) \leq c(\underline{\pi}(b))+2\ell\phi$ implies $P(b) < c(\underline{\pi}(b))+h(\phi) < c(\underline{\pi}(b))+\frac{\varepsilon}{2}$ and so, $K > 1$. By Lemma 10, for all b, $P(b) > \frac{1}{2}(y^*(0,0)+c(0))$ for sufficiently large δ. This implies that $b-(K-1)\phi > \eta$ for sufficiently large δ, as $\phi < \frac{\xi}{2\ell}$. For all $2 \leq k < K$, $P(b-k\phi) > c(\underline{\pi}(b-(k-1)\phi)) + 2\ell\phi$ and so by Lemma 8,

$$P(b-(k-1)\phi) > P(b-(k-2)\phi) - f(\phi,\delta). \tag{29}$$

Therefore,

$$\begin{aligned} P(b-(K-1)\phi) &> P(b)-(K-1)f(\phi,\delta) \\ &\geq c(\underline{\pi}(b))+\varepsilon-(K-1)f(\phi,\delta) \\ &> c(\underline{\pi}(b-(K-1)\phi))+\varepsilon-f(\phi,\delta)/\phi \\ &> c(\underline{\pi}(b-(K-1)\phi))+\tfrac{\varepsilon}{2}, \end{aligned}$$

where the first inequality is by iterating (29), the second by $P(b) \geq c(\underline{\pi}(b))+\varepsilon$, the third by the monotonicity of c and $K \leq 1/\phi$, the last is by the choice of ϕ. On the other hand, by Lemma 10, $P(b-(K-1)\phi) < c(\underline{\pi}(b-(K-1)\phi)) + h(\phi) < c(\underline{\pi}(b-(K-1)\phi)) + \frac{\varepsilon}{2}$. Therefore, no such K exists which implies that the inequality (29) holds for all k up to $K^* = 1+\lfloor {}^{b-\eta}/_{\phi}\rfloor$. Summing

(29) for all k,

$$\begin{aligned} P(b) &< P(b-(K^*-1)\phi)+(K^*-1)f(\phi,\delta) \\ &< P(\eta)+f(\phi,\delta)/\phi \\ &< P(\eta)+\tfrac{\varepsilon}{2}. \end{aligned}$$

By the last statement in Lemma 13 in the Online Appendix, $P^*(b) = y^*(0,0)$ for $b \in [0,\eta]$. Therefore, $P(b) < y^*(0,0)+\varepsilon$ which gives the desired conclusion. □

Proof of Theorem 1. The continuation utility of type s of the seller in the seller punishing equilibrium is bounded above by $P(\overline{\pi}(s)) - c(s)$. By Lemma 11, $\lim_{\delta\to 1} \max_{s\in[0,1]} |P(\overline{\pi}(s)) - c(s) - \max\{y^*(0,0) - c(s), 0\}| = 0$ which gives the desired conclusion. □

A.2 Proofs for Section 4.1

Proof of Lemma 2. Denote the uniform distribution of types on Ω_η by F and note that F is affiliated. Let $F(s|b) = \frac{\max\{\min\{s,\overline{\pi}(b)\}-\underline{\pi}(b),0\}}{\overline{\pi}(b)-\underline{\pi}(b)}$ be the c.d.f. of the buyer type b's beliefs and $f(s|b)$ be the corresponding p.d.f. We show that if $b_\cdot$ satisfies the equation (5), then it is a best response to the threshold strategy $s_\cdot$. Buyer's type b chooses the acceptance time t to maximize $u(b,t)$ given by

$$u(b,t) = \int_0^t e^{-ru}(v(b)-q^B)dF(s_u|b) + (1-F(s_t|b))e^{-rt}(v(b)-q^S).$$

The first-order condition for this problem is

$$(q^S - q^B)f(s_t|b)\dot{s}_t = r(v(b)-q^S)(1-F(s_t|b)) \tag{30}$$

from which it immediately follows that $|b_t - s_t| \leq \eta$ (otherwise, the left-hand side is zero) and threshold type b_t strictly prefers to accept at time t. This implies that $b_0 - b_\infty < 2\eta$ and $s_\infty - s_0 < 2\eta$, which in turn together with $b_\infty = s_\infty + \eta$ implies that the probability of a positive delay is at most $\left(\frac{1}{2}(4\eta)^2 - \frac{1}{2}(2\eta)^2\right)/\eta(2-\eta) = \frac{6\eta}{2-\eta} \leq 6\eta$. (Note that the density of F on Ω_η is given by $\frac{1}{1-(1-\eta)^2} = \frac{1}{\eta(2-\eta)}$).

From the first-order condition (30),

$$\begin{aligned} u(1,t(1)) - u(\tilde{b},t(\tilde{b})) &= \int_{\tilde{b}}^1 \left(\frac{\partial}{\partial b}u(b,t(b)) + \frac{\partial}{\partial t}u(b,t(b))t'(b)\right)db \\ &= \int_{\tilde{b}}^1 \frac{\partial}{\partial b}u(b,t(b))db, \end{aligned} \tag{31}$$

where $t(b)$ is the inverse of b_t. In Claim 2 below, we show that $u(b,t)$ satisfies the smooth single crossing difference (SSCD) condition in $(b, -t)$. Together with the envelope formula (31), this verifies the conditions of Theorem 4.2 in Milgrom (2004) and proves that b_t is a best response to s_t. Therefore, (b_t, s_t) constitute a BNE of $\mathcal{G}(q^S, q^B)$.

Claim 2. $u(b,t)$ satisfies the SSCD condition in $(b,-t)$ for $b \in (b_\infty, b_0)$ and $t \in (0,\infty)$.

Proof: We will show the following conditions are satisfied which imply the SSCD.

1. $u(b,t)$ satisfies the (strict) single crossing difference condition in $(b,-t)$, i.e. for all $\tilde{t} > t$ and $\tilde{b} > b$,
$$u(b,t) - u(b,\tilde{t}) \geq 0 \implies u(\tilde{b},t) - u(\tilde{b},\tilde{t}) > 0.$$
2. for all t, if $\frac{\partial}{\partial t}u(b,t) = 0$, then for all $\delta > 0$, $\frac{\partial}{\partial t}u(b,t-\delta) \geq 0$ and $\frac{\partial}{\partial t}u(b,t+\delta) \leq 0$.

Let us start with the single crossing difference condition. Consider $b < \tilde{b}$ and $t < \tilde{t} \leq T$ and suppose that
$$u(b,t) \geq u(b,\tilde{t}). \tag{32}$$
We will show that $u(\tilde{b},t) > u(\tilde{b},\tilde{t})$. Define function
$$g(u|b,t) = e^{-ru}(v(b) - q^B)1\{u < t)\} + e^{-rt}(v(b) - q^S)1\{u \geq t\}.$$
Then
$$\int_0^T g(u|b,t)dF(s_u|b) \geq \int_0^T g(u|b,\tilde{t})dF(s_u|b) \geq \int_0^T g(u|b,\tilde{t})dF(s|\tilde{b}),$$
where the first inequality follows from (32), the second inequality follows from the fact that $g(\cdot|b,\tilde{t})$ is decreasing and $F(\cdot|\tilde{b})$ first-order stochastically dominates $F(\cdot|b)$ (as F is affiliated). This implies that
$$u(b,t) = \int_0^t e^{-ru}(v(b) - q^B)dF(s_u|b) + (1 - F(s_t|b))e^{-rt}(v(b) - q^S)$$
$$\geq \int_0^{\tilde{t}} e^{-ru}(v(b) - q^B)dF(s_u|\tilde{b}) + (1 - F(s_{\tilde{t}}|\tilde{b}))e^{-rt}(v(b) - q^S),$$
or equivalently,
$$v(b)\left(\int_0^t e^{-ru}dF(s_u|b) + (1 - F(s_t|b))e^{-rt} - \int_0^{\tilde{t}} e^{-ru}dF(s_u|\tilde{b}) - (1 - F(s_{\tilde{t}}|\tilde{b}))e^{-r\tilde{t}}\right)$$
$$\geq q^S - \int_0^{\tilde{t}} e^{-ru}q^B dF(s_u|\tilde{b}) - (1 - F(s_{\tilde{t}}|\tilde{b}))e^{-r\tilde{t}}q^S. \tag{33}$$
We will show that the left-hand side of (33) is positive and so, the left-hand side would increase

if we substitute $v(\tilde{b})$ instead of $v(b)$. This in turn implies that $u(\tilde{b},t) > u(\tilde{b},\tilde{t})$ and completes the proof of the strict single crossing difference. Let $h(u|t) = e^{-ru}1\{u < t)\} + e^{-rt}1\{u \geq t\}$. Then the left-hand side of (33) is equal to

$$\begin{aligned}
&v(b)\left(\int_0^T h(u|t)dF(s_u|b) - \int_0^T h(u|\tilde{t})dF(s_u|\tilde{b})\right)\\
\geq &v(b)\left(\int_0^T h(u|t)dF(s_u|\tilde{b}) - \int_0^T h(u|\tilde{t})dF(s_u|\tilde{b})\right)\\
= &v(b)\int_0^T (h(u|t) - h(u|\tilde{t}))dF(s_u|\tilde{b}) > 0,
\end{aligned}$$

where the first inequality follows from $F(\cdot|\tilde{b})$ first-order stochastically dominates $F(\cdot|b)$ and $h(\cdot|t)$ decreasing, and the last term is strictly positive by $t < \tilde{t}$.

Now, let us show the second requirement of the SSCD condition. Suppose $\frac{\partial}{\partial t}u(b,t) = 0$. By taking the partial derivative

$$e^{rt}\frac{\partial}{\partial t}u(b,t) = (q^S - q^B)f(s_t|b)\dot{s}_t - r(v(b) - q^B)(1 - F(s_t|b)),$$

we get that

$$\begin{aligned}
&e^{rt}\frac{\partial}{\partial t}u(b-\delta,t) =\\
&\qquad (q^S - q^B)f(s_t|b-\delta)\dot{s}_t - r(v(b-\delta) - q^S)(1 - F(s_t|b-\delta)) =\\
&\qquad (1 - F(s_t|b-\delta))\left((q^S - q^B)\frac{f(s_t|b-\delta)}{1 - F(s_t|b-\delta)}\dot{s}_t - (r(v(b-\delta) - q^S)\right).
\end{aligned}$$

Since $v(b-\delta) \leq v(b)$ and $\frac{f(s_t|b-\delta)}{1-F(s_t|b-\delta)} \geq \frac{f(s_t|b)}{1-F(s_t|b)}$ (by the affiliation of f), it follows that $\frac{\partial}{\partial t}u(b-\delta,t) \geq 0$. Showing that $\frac{\partial}{\partial t}u(b+\delta,t) \leq 0$ is analogous. *q.e.d.* □

The following lemma (proven in the Online Appendix) is the key mathematical fact in the proof of Lemma 3.

Lemma 12. *Consider $b_\infty \in (0, 1-\eta)$, $s_\infty = b_\infty + \eta$, q^B, q^S that satisfy*

$$\max\{c(s_\infty), y^*(0,0)\} < q^B < q^S < \min\{v(b_\infty), y^*(1,1)\}. \tag{34}$$

There exists $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$ there are positive sequences $(x_k, y_k)_{k=1}^\infty$ that *satisfy the recursive system*

$$\begin{cases} x_{k+1} = (1-\alpha^B(y_{k+1}))x_k - \alpha^B(y_{k+1})y_{k+1}, \\ y_{k+1} = (1-\alpha^S(x_k))y_k - \alpha^S(x_k)x_k, \\ b_\infty + x_k \le s_\infty - y_k + \eta; \end{cases} \tag{35}$$

where

$$\begin{aligned} \alpha^B(y) &= \frac{(1-\delta^2)(q^B - c(s_\infty - y))}{\delta(q^S - c(s_\infty - y)) - \delta^2(q^B - c(s_\infty - y))}, \\ \alpha^S(x) &= \frac{(1-\delta^2)(v(b_\infty + x) - q^S)}{\delta(v(b_\infty + x) - q^B) - \delta^2(v(b_\infty + x) - q^S)}. \end{aligned}$$

Proof of Lemma 3. We construct on-path threshold acceptance strategies. Types of the buyer above b_n and types of the seller below s_n accept the opponent's offer in even round n for the buyer and odd round n for the seller. Otherwise, players make counter-offers q^S for the seller and q^B for the buyer. By Lemma 12, we can construct sequences of threshold types b_n and s_n so that corresponding sequences x_k and y_k defined by $x_k = b_{2k} - b_\infty$ and $y_k = s_\infty - s_{2k-1}$ for $k = 1, 2, \ldots$ satisfy (35). Since (x_k, y_k) is a positive trajectory and $\alpha^B(y) > 0$ whenever $y > 0$, from (35) it follows that $x_{k+1} - x_k = -\alpha^B(y_{k+1})(x_k + y_{k+1}) < 0$ for all $n \in \mathbb{N}$, and analogously, $y_{k+1} - y_k < 0$. Hence, b_n and s_n are monotone sequences. Since (x_k, y_k) converges to $(0, 0)$, the limits of b_n and s_n are b_∞ and s_∞, respectively. The form of functions $\alpha^B(x)$ and $\alpha^S(y)$ implies that

$$v(b_n) - q^S = \delta\alpha_n^S\left(v(b_n) - q^B\right) + \delta^2\left(1 - \alpha_n^S\right)\left(v(b_n) - q^S\right) \text{ for } n \text{ even}, \tag{36}$$

$$q^B - c(s_n) = \delta\alpha_n^B\left(q^S - c(s_n)\right) + \delta^2\left(1 - \alpha_n^B\right)\left(q^B - c(s_n)\right) \text{ for } n \text{ odd}, \tag{37}$$

where

$$\alpha_n^S = \frac{s_{n+1} - s_{n-1}}{\overline{\pi}(b_n) - s_{n-1}}, \tag{38}$$

$$\alpha_n^B = \frac{b_{n-1} - b_{n+1}}{b_{n-1} - \underline{\pi}(s_n)}. \tag{39}$$

By the same argument as in Lemma 2 we can show threshold strategies b_n and s_n are optimal on-path.

All deviations from acceptance strategies b_n and s_n are ignored. To deter deviations from offers q^B and q^S specify that after deviations from price offers q^B and q^S, players switch to the punishing equilibrium of the deviator. By Theorem 1, in such an equilibrium the expected utility of the deviator is uniformly (over all types of the deviator) close to the reservation utility as δ converges to one. On the other hand, by following the equilibrium strategy any seller type $s \leq \overline{\pi}(b_0)$ gets at least $q^B - c(s)$, and any buyer type $b \geq \underline{\pi}(s_0)$ gets at least $v(b) - q^S$. These utilities are bounded away from the reservation utility by $\max\{c(s_\infty), y^*(0,0)\} < q^B < v(0)$ and $c(1) < q^S < \min\{v(b_\infty), y^*(1,1)\}$. This proves that the constructed thresholds constitute the PBE. Sequences b_n and s_n can be linearly extrapolated to continuous time and they converge to b_t and s_t, resp. This implies the convergence of PBE outcomes to $\nu^p(s_\infty, q^S, q^B)$ as $\delta \to 0$. $\square$

Proof of Lemma 4. For all $z = 1, \ldots, Z-1$, let $\hat{s}^z = b^z - \eta$ and $\hat{q}_z$ be such that the seller type $\hat{s}_z$ is indifferent between having his offer $\hat{q}_z$ accepted in the current round and accepting q_z in the next round, i.e. $\hat{q}_z - c(\hat{s}^z) = \delta(q_z - c(\hat{s}^z))$. Denote $\hat{s}^0 = 0, \hat{s}^Z = 1$, and $\hat{q}_Z = q_Z$.

For any $z = 1, \ldots, Z-1$, suppose that only types of the seller in $[\hat{s}^z, \hat{s}^{z+1}]$ and of the buyer in $[b^z - \eta, b^{z+1}]$ remain in the game. We can follow the proof of Lemma 3, to construct for sufficiently large δ the continuation PBE with constant offers on the equilibrium path q_z for the buyer and $\hat{q}_{z+1}$ for the seller, and acceptance threshold strategies b_n^z and s_n^z such that $\lim_{n\to\infty} b_n^{z-1} = b^z$ and $\lim_{n\to\infty} s_n^z = s^z$. In this PBE, $b_2^z - b^z < 2\eta$ and $s^z - s_3^z < 2\eta$. Since Z is finite, for sufficiently large δ the constructed strategies constitute continuation PBEs for all $z = 1, \ldots, Z-1$. Let $\hat{b}^z = b_2^z$ for $z = 1, \ldots, Z-1$ and $\hat{b}^0 = 0$, $\hat{b}^Z = 1$.

The equilibrium strategies are described as follows. In the first round, for $z = 1, \ldots, Z$ seller types in $[\hat{s}^{z-1}, \hat{s}^z]$ make offer $\hat{q}_z$. Since $\hat{s}^z - \hat{s}^{z-1} > b^z - b^{z-1} > 4\eta$ for $z = 2, \ldots, Z$, each type of the buyer expects to receive one of at most two offers in the first round on-path. In the second round, for $z = 1, \ldots, Z-1$, buyer types in $[\hat{b}^{z-1}, \hat{b}^z]$ accept offer $\hat{q}_z$, but reject offer $\hat{q}_{z+1}$ and make a counter-offer q_z. Buyer types in $[\hat{b}^{Z-1}, \hat{b}^Z]$ accept $\hat{q}_Z$. Starting from the third round, if in the first two rounds the buyer offered q_z and the seller offered $\hat{q}_{z+1}$ for some z, then the buyer continues offering q_z and the seller continues offering $\hat{q}_{z+1}$ and players follow acceptance strategies b_n^z and s_n^z constructed above.

Denote the set of seller's on-path offers by $Q^S = \{\hat{q}_1, \ldots, \hat{q}_Z\}$ and the set of buyer's on-path offers by $Q^B = \{q_1, \ldots, q_{Z-1}\}$. After any deviation to offers not in Q^S for the seller and not in Q^B for the buyer, players switch to the punishing continuation equilibrium of the deviator constructed in Theorem 1. If the seller deviates from the equilibrium path to a lower price offer in Q^S, then such an offer is accepted by all remaining buyer types. If the seller deviates to a higher price offer in Q^S, then such an offer is rejected by all remaining buyer types and in subsequent rounds players return to following on-path acceptance strategies b_n^z and s_n^z and offers q_z and $\hat{q}_{z+1}$. Same happens if the seller deviates from the acceptance strategy. Strategies after buyer's deviations from on-path strategies are defined analogously.

Let us verify that constructed strategies constitute the PBE for sufficiently large δ. By construction, after first two rounds the continuation strategies constitute PBEs. By the choice of $\hat{q}_z$ and q_{z+1} and Theorem 1, no player prefers to deviate in the first two rounds from the equilibrium price offers to offers outside Q^S and Q^B for δ sufficiently large. Since $q_{z-1} < q_z$

for all z, a deviation to a lower offer in Q^S is worse than accepting the buyer's price offer for sufficiently large δ. A deviation to a higher offer in Q^S in one round will be rejected for sure for sufficiently large δ, as the buyer expects that the seller will return to the equilibrium path and decrease her price offer in the next round. Thus, such a deviation is not profitable either and so, the seller does not have incentives to deviate to other offers in Q^S for sufficiently large δ. (The argument for the buyer is symmetric). Finally, since $\hat{b}^{z-1} > b^{z-1} = \hat{s}^{z-1} + \eta$, all buyer types in $B_{\hat{s}^z}$ accept offer $\hat{q}_z$, but reject q_{z+1}. By $\hat{q}_z - c(\hat{s}^z) = \delta(q_z - c(\hat{s}^z))$, seller type $\hat{s}^z$ is indifferent between offering $\hat{q}_z$ that is accepted for sure and offering q_{z+1} that is rejected for sure and accepting the buyer's counter-offer q_z. By the single-crossing property of payoffs, seller types above $\hat{s}^z$ strictly prefer to accept the offer q_z in two rounds, and seller types below $\hat{s}^z$ strictly prefer that the buyer accepts $\hat{q}_z$ in the next round.

By construction, on the equilibrium path, types of the buyer in $[b^z, b^{z+1}]$ and types of the seller in $[s^z, s^{z+1}]$ trade only at price q_z or $\hat{q}_{z+1}$ where the latter converges to q_{z+1} as $\delta \to 1$. The positive delay occurs only when players offer q_z and $\hat{q}_{z+1}$ in the beginning. For each z, the probability of such initial offers is at most 6η by Lemma 3, and so the probability of positive delay is at most $6\eta(Z-1)$. □

Proof of Theorem 2. Fix $\gamma, \beta \in (0,1)$ and choose $b^z = b^{z-1} + \sqrt{\eta}$ and $q_z = q(b^z, \gamma, \beta)$ for $z = 1, \ldots, Z$ and $Z = \left\lfloor \frac{1}{\sqrt{\eta}} \right\rfloor$. Observe that for $\gamma, \beta \in (0,1)$, $q(\cdot, \gamma, \beta)$ is strictly increasing with the derivative bounded from above and below by some positive constants C and c, resp., and for all ι, $c(\iota) < q(\iota, \gamma, \beta) < v(\iota)$ and $q(\iota, \gamma, \beta) \in (y^*(0,0), y^*(1,1))$. Thus, for sufficiently small η, the conditions of Lemma 4 are satisfied and there is a frequent-offer PBE limit (τ_η, ρ_η) such that types of the buyer in $[b^z, b^{z+1}]$ and types of the seller in $[s^z, s^{z+1}]$ trade at price q_z or q_{z+1} and probability of a positive delay is at most $6\eta(Z-1) \sim \frac{1}{\sqrt{\eta}}$. Therefore, $\mathbb{P}_\eta\left(\tau_\eta(s,b) > 0 \text{ or } |\rho_\eta(s,b) - q(s,\gamma,\beta)| > c_0\eta\right) = \frac{c_1}{\sqrt{\eta}}$ for some constants c_0 and c_1. Therefore, $(0, q(b, \gamma, \beta))$ is an almost-public information limit.

To characterize the Pareto frontier, fix $\beta \in (0,1)$ such that $U^S = \mathbb{E}_{(s,b)}[q(s,1,\beta) - c(s)]$. Fix $\varepsilon > 0$. By setting $q_z = q(b^z, 1-\varepsilon, \beta(1-\varepsilon) + \frac{1}{2}\varepsilon)$ in the construction above, we can construct for any $\varepsilon > 0$ a sequence of frequent-offer PBE limit $(\tau_{\eta,\varepsilon}, \rho_{\eta,\varepsilon})$ that converges to $(0, q(b, 1-\varepsilon, \beta(1-\varepsilon) + \frac{1}{2}\varepsilon))$ as $\eta \to 0$. By taking the diagonal subsequence in ε and η, we get the sequence of frequent-offer PBE limit that converges to $(0, q(b,1,\beta))$ as $\eta \to 0$. Similarly, by setting $q_z = q(b^z, \varepsilon, \frac{1}{2})$ we can show that the complete-information outcome is an almost-public information limit. □

A.3 Proofs for Section 4.2

Proof of Theorem 5. Since $s_t < s_T = b_T - \eta < \underline{\pi}(b_t)$ for $t < T$, type b_t assigns probability zero to his offer being accepted before T. Type b_t chooses the acceptance time t to maximize

$e^{-rt}(v(b_t) - q_t^S)$ for which the first-order condition is given by the equation (8) (equation (9) for threshold seller types). The sufficiency of the first-order condition follows from the single-crossing property of payoffs. □

Proof of Theorem 3. We construct PBEs in grim-trigger strategies. Players start the game by following the *main path* and continue following it so long as there were no deviations in the past. If one of the sides detects a deviation, then the play switches to the *punishing path of the deviator*.

Construction of the main path $(\tilde{b}_n, \tilde{s}_n, \tilde{q}_n^B, \tilde{q}_n^S)$**:** We consider the following on-path strategies. In any odd round n, types of the seller in $[\tilde{s}_{n-1}, \tilde{s}_n]$ make an offer $\tilde{q}_n^B$ which is accepted by the buyer and all remaining types make an unacceptable offer $v(1)$. In any even round n, types of the buyer in $[\tilde{b}_n, \tilde{b}_{n-1}]$ make an offer $\tilde{q}_n^S$ which is accepted by the seller and all remaining types make an unacceptable offer $c(0)$.

Consider a tuple $(b_t, s_t, q_t^B, q_t^S, T)$ such that $q_t^S = \frac{1}{2}(v(b_t - \frac{\eta}{2}) + c(b_t - \frac{\eta}{2}))$ and $q_t^B = \frac{1}{2}(v(s_t + \frac{\eta}{2}) + c(s_t + \frac{\eta}{2}))$, and b_t and s_t so that they solve the system $(8) - (9)$ with the initial condition $b_0 = 1$ and terminal condition $b_T = s_T + \eta$. We construct the discrete-time approximation of s_t and b_t using the Euler method. Suppose $N = \lfloor \frac{T}{\Delta} \rfloor$ is even and let $\tilde{s}_{N+1} = 1$,$\tilde{s}_N = s_T$, $\tilde{b}_N = b_T$. For odd $n < N$, $\tilde{s}_n = \tilde{s}_{n+1} - \dot{s}_{(n+1)\Delta}\Delta$ and $\tilde{b}_n = \tilde{b}_{n+1}$, and for even $n < N$, $\tilde{b}_n = \tilde{b}_{n+1} - \dot{b}_{(n+1)\Delta}\Delta$ and $\tilde{s}_n = \tilde{s}_{n+1}$. We construct paths $\tilde{q}_n^S$ (for n odd) and $\tilde{q}_n^B$ (for n even) backwards in time starting from N and $\tilde{q}_{N+1}^S = q_N^B = q_T^S$ as follows: for $n \leq N$,

$$v(\tilde{b}_n) - \tilde{q}_n^S = \delta^2(v(\tilde{b}_n) - \tilde{q}_{n+2}^S), \tag{40}$$

$$\tilde{q}_n^B - c(\tilde{s}_n) = \delta^2(\tilde{q}_{n+2}^B - c(\tilde{s}_n)). \tag{41}$$

$(\tilde{b}_n, \tilde{s}_n, \tilde{q}_n^B, \tilde{q}_n^S)$ converges uniformly to (b_t, s_t, q_t^B, q_t^S) which implies the convergence of the PBE outcomes (τ, ρ) to $(\tau_\eta^\dagger, \rho_\eta^\dagger)$. By construction of paths q_t^S and q_t^B, in the war-of-attrition game $\mathcal{G}(q_t^S, q_t^B)$, the expected continuation utility at any time $t \leq T$ for every type is greater than the lower bound in Lemma 1.

Construction of the punishing path: We consider the punishment of the seller and the strategies are symmetric for the punishment of the buyer. The following seller's deviations are possible:

- If the seller makes in round n an offer different from $v(1)$ and $\tilde{q}_n^B$, the play switches to the continuation equilibrium with optimistic conjectures of the buyer. By Theorem 1, such a deviation is not profitable.

- If type $s \in [\tilde{s}_{n-1}, \tilde{s}_n)$ of the seller mimics a lower type, then she reveals herself with an offer lower than $\tilde{q}_n^B$. We specify equilibrium strategies so that such an offer is accepted by the buyer no matter whether the buyer detects such offer as a deviation or not and if it is rejected, then the subsequent play returns to the main path. By the single-crossing property of payoffs, such a deviation is not profitable for type s. Since the play returns to the main path in case the deviating offer is rejected, the buyer prefers to accept such an offer which is lower than subsequent on-path offers.

- If type $s \in [\tilde{s}_{n-1}, \tilde{s}_n)$ mimics a higher type, then she does not make any acceptable offers until some round $n' > n$. Such a deviation is detected by the buyer only when $b < \underline{\pi}(\tilde{s}_{n'})$. Specify equilibrium strategies as follows. In round n', type s makes offer $\tilde{q}^B_{n'}$ which is accepted by types of the buyer who did not detect the deviation. Types of the buyer who detected the deviation immediately switch to optimistic conjectures. Then they wait until the seller offers $\tilde{q}^B_{n'}$ and make a counter-offer $\underline{y}(0,0)$ and the play switches to the continuation equilibrium with optimistic conjectures of the buyer. If the buyer deviates and makes some offer in round before n', then the seller switches to optimistic conjectures. By the single-crossing property of payoffs and construction of offers in (40), even if the buyer never detects the deviation, type s prefers to reveal herself in round n rather than later. On top of that delaying the revelation increases the probability that the buyer detects the deviation which makes longer delay even more costly. Hence, it is optimal for the seller to reveal herself as soon as possible.

  The type of the buyer who detected the deviation prefers to offer $c(0)$ and wait for the seller to reveal the deviation in round n'. Indeed, by Theorem 1, for sufficiently large δ, this guarantees an immediate trade at a price close to the minimal price $\max\{y^*(0,0), c(\underline{\pi}(b))\}$, while if she deviates and makes an off-path offer, then both sides hold optimistic conjectures and by Lemma 7, the buyer expects a price of trade close to $y^*(\underline{\pi}(b), b)$ for δ sufficiently large.

The convergence of $\rho^\dagger_\eta$ to $y^*(s,b)$ as $\eta \to 0$ follows from the construction of q^S_t and q^B_t and acceptance strategies b_t and s_t, and the positive expected delay $\mathbb{E}_\eta[\tau^\dagger_\eta]$ follows from the fact that the left-hand side of (8) and (9) is bounded (and so, the acceptance is gradual). □

Proof of Theorem 4. By Lemma 6, for any $\varepsilon_0 > 0$ and $K(b,\eta) = \lfloor \varepsilon_0/2\eta \rfloor$,

$$\begin{aligned} P_b &\leq (1-\delta) \sum_{k=0}^{K(b,\eta)} \delta^{2k}(v(b+\varepsilon) + \delta c(b+\varepsilon)) + \delta^{2K(b,\eta)} y(1,1) \\ &\leq (1-\delta) \sum_{k=0}^{K(b,\eta)} \delta^{2k}(v(b) + \delta c(b)) + 2(1-\delta)\varepsilon_0 \ell + \delta^{2K(b,\eta)} y(1,1) \end{aligned}$$

As $\eta \to 0$, the last term goes to zero while the first term converges to $\overline{y}(b,b)$. Since ε_0 was chosen arbitrarily, $\lim_{\eta\to 0} P_b \leq \overline{y}(b,b)$ and by the same argument, $\lim_{\eta \to 0} p_s \geq \underline{y}(s,s)$. Players' types are within distance η and so, for all s and b drawn from Ω_η, $|\overline{y}(b,b) - \underline{y}(s,s)| < \ell\eta$. This implies that for some constants c_0, c_1, $|\rho_\delta(s,b) - y^*(s,b)| < c_0\eta$ and $\tau_\delta < c_0\eta$ which gives the conclusion of the theorem. $\square$

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Online Appendix (Not for Publication)

Existence

In this subsection, we prove the existence and derive several useful properties of the punishing equilibrium.

Theorem 5. *For all δ sufficiently close to 1, there exists the punishing equilibrium.*

The key element of the proof of theorem 5 is the existence of functions $t(\cdot,\cdot)$ and $P(\cdot)$ that satisfy (2) and (11). The proof of the existence of such functions is constructive and the construction is carried out starting from the bottom of the type distribution.

We first analyze strategies of type 0 of the seller and types of the buyer in $[0,\eta]$. Such types of the buyer put probability one on the type 0 of the seller, and the model is reduced to the model with one-sided incomplete information and alternating offers. The following result is standard in the literature (see Grossman and Perry (1986), Gul and Sonnenschein (1988)).[24]

Lemma 13. *For all δ sufficiently close to 1, there exists a perfect Bayesian equilibrium in a game between type 0 of the seller and types of the buyer in $[0,\eta]$, in which on-path (a) the buyer makes price offer $\underline{y}(0,0)$ and accepts offers according to the right-continuous and strictly increasing willingness to pay function P^0; (b) there exists $\bar{\beta} \in [0,\eta]$ such that if the highest remaining buyer type is below $\bar{\beta}$, then type 0 of the seller accepts offer $\underline{y}(0,0)$; (c) given the highest remaining type of the buyer $\beta \in (\bar{\beta},\eta]$, type 0 of the seller allocates to all types above $t(\beta,0)$ in the current round. Moreover, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) < 1$ such that for all $\delta > \delta(\varepsilon)$, the first price offer of type 0 of the seller does not exceed $y^*(0,0)+\varepsilon$.*

Proof. The proof is standard and we only sketch the argument. Following Fudenberg et al. (1985), we can construct a perfect Bayesian equilibrium in a game between type 0 of the seller and types of the buyer in $[0,\eta]$ in which the buyer is restricted to either accept the seller's offer or make the counter-offer $\underline{y}(0,0)$.[25] Such equilibrium takes the form as described in the statement of the lemma. By the argument from the Theorem 3 in Gul et al. (1986), the Coase Conjecture

[24] One detail worth mentioning is that despite the fact that in the punishing equilibrium, type 0 of the seller follows a pure strategy on the equilibrium path, off-path mixing might be necessary (see footnote 14). This possibility can easily be incorporated into the analysis. For notation simplicity, we will assume that the seller screening strategy in Lemma 13 is pure.

[25] The argument in Fudenberg, Levine, and Tirole (1985) should be slightly modified. As in their paper, we start by showing that for β smaller than some $\bar{\beta}$ the seller prefers to accept $\underline{y}(0,0)$ rather than continue the screening. This implies that there is a finite date after which bargaining ends with probability one by the argument analogous to Lemma 3 in Fudenberg, Levine, and Tirole (1985). we follow the steps in their proof of Proposition 1 to construct equilibrium strategies by backward induction on beliefs starting from beliefs supported by $[0,\beta], \beta < \bar{\beta}$ with the only difference that instead of asking price $v(0)$, the seller accepts price offer $\underline{y}(0,0)$ for such beliefs.

holds for such game, and for any $\varepsilon > 0$, after any history the first price offer of the seller does not exceed $y^*(0,0) + \varepsilon$ for δ sufficiently close to one.

To support the constructed equilibrium as an equilibrium in the game with unrestricted counter-offers of the buyer specify the following punishment for deviations of the buyer. If the buyer makes an offer different from $\underline{y}(0,0)$, then the seller puts probability one on type η of the buyer and the game proceeds as in the unique subgame perfect equilibrium of the game with complete information with the seller cost equal $c(0)$ and the buyer valuation equal $v(\eta)$. Then trade happens almost immediately at a price that is close to $y^*(0,\eta)$ for δ close to one. On the other hand, by the Coase Conjecture, the first offer of the seller is close to $y^*(0,0) < y^*(0,\eta)$ for δ close to one, making the deviation of the buyer non-profitable. $\square$

Strategies for the rest of the types are constructed via the following iterative algorithm. Let $c(\eta,\delta) > 0$ be a constant specified in Lemma 17 below, I be the smallest integer such that $Ic(\eta,\delta) \geq 1-\eta$ and denote $b_i \equiv \eta + ic(\eta,\delta)$ for $i = 1,\dots,I$.

Iterative Algorithm

Input: Define $\pi^0(b) = \begin{cases} P^0(b), \text{ for } b \in [0,\eta], \\ v(b), \text{ for } b \in (\eta,1]. \end{cases}$

Execute Step $i, i = 1,\dots,I+1$**.**

Step i**:** Construct a screening policy $\tau^i(\beta,s)$ that is a best-reply to $\pi^{i-1}(b)$. Construct $\pi^i(b)$ by

$$\pi^i(b) = \begin{cases} \pi^{i-1}(b), & \text{for } b \in [0,b_{i-1}], \\ (1-\delta^2)v(b) + \delta^2\pi^{i-1}(\tau^i(\underline{\pi}(b))), & \text{for } b \in (b_{i-1},b_i], \\ v(b), & \text{for } b \in (b_i,1]. \end{cases}$$

Output: $P(b) = \pi^{I+1}(b)$, $t(\beta,s) = \tau^{I+1}(\beta,s)$.

I next prove several auxiliary results that allow me to verify that the iterative algorithm produces the punishing equilibrium for sufficiently large δ.

Lemma 14. *Suppose $t(\beta,s)$ is a best-reply to willingness to pay P. Then (a) $R(\beta,s)$ is non-decreasing in β; (b) for $0 \leq \beta'' < \beta' \leq 1$, it holds $0 < R(\beta',s) - R(\beta'',s) \leq \Sigma(\beta'-\beta'')$ whenever $R(\beta',s) > 0$, and $R(\beta',s) = R(\beta'',s) = 0$ whenever $R(\beta',s) = 0$; (c) R is Lipschitz-continuous in both β and s of modulus $\ell_R = \ell + \Sigma$.*

Proof. Parts *(a)* and *(b)* follow from Lemma A.2 in Ausubel, Deneckere (1989). To show that R is Lipschitz-continuous, consider two types s and s'. Let $R(\beta,s,s')$ be the value function of type s from following the screening policy $t(\beta,s')$ when the highest remaining type of the buyer is

β.[26] Let $p_s^{s'}$ and $q_s^{s'}$, respectively, be the expected discounted transfer and discounted probability of allocation, respectively, both multiplied by the mass of remaining types according to type s when type s follows the screening policy of type s'. Then

$$\begin{aligned} R(\beta,s) &\geq R(\beta,s,s') \\ &= p_s^{s'} - q_s^{s'}c(s) \\ &\geq p_s^{s'} - q_s^{s'}c(s') - |c(s)-c(s')| \\ &\geq p_{s'}^{s'} - q_{s'}^{s'}c(s') - (\ell+\Sigma)|s-s'| \\ &= R(\beta,s') - (\ell+\Sigma)|s-s'|. \end{aligned}$$

The equalities are by the definition of $R(\beta,s,s')$ and $R(\beta,s')$. The first inequality follows from the fact that type s prefers policy $t(\beta,s)$ to $t(\beta,s')$. The second inequality is by $q_s^{s'} \in [0,1]$. To see the last inequality, consider two cases. When $s > s'$, by using the screening policy $t(\beta,s')$, type s gets the same profit from types of the buyer in $[\underline{\pi}(s),\overline{\pi}(s')]$ as type s', but looses at most Σ by not screening types in $[\underline{\pi}(s'),\underline{\pi}(s))$ that have mass at most $s-s'$ (as she assigns probability one to them). When $s < s'$, by using the screening policy $t(\beta,s')$, type s gets the same profit from types of the buyer in $[\underline{\pi}(s'),\overline{\pi}(s)]$ as type s', but looses at most Σ from types of the buyer in $[\overline{\pi}(s),\overline{\pi}(s')]$ (as she assigns probability one to them). By reversing the roles of s and s' and repeating the argument, we get $|R(\beta,s)-R(\beta,s')| \leq (\ell+\Sigma)|s-s'|$ which proves the Lipshitz-continuity. □

Lemma 15. *Suppose screening policy t is a best-reply to willingness to pay P. Then (a) t is non-decreasing in β and s; (b) for any β, $T(\beta,\cdot)$ has a closed graph and $t(\beta,\cdot)$ is right-continuous in s.*

Proof. Part (b) follows from the generalization of Theorem of the Maximum in Ausubel and Deneckere (1988). Part (a) is an analogue of Proposition 2 in Hopenhayn and Prescott (1992). Due to the special functional form, we can prove it under weaker complementarity conditions that are satisfied in my environment. Let $\pi(\beta,s,b) \equiv (\beta - \min\{\beta,b\})\max\{0,(P(b)-c(s))\}$. The Bellman equation (11) can be written as $R(\beta,s) = \max_{b\in B_s}\{\pi(\beta,b,s)+\delta^2 R(b,s)\}$. Consider the functional equation $\mathbb{T}R(\beta,s) = \max_{b\in B_s}\{\pi(\beta,b,s)+\delta^2 R(b,s)\}$. we show that $\mathbb{T}R$ has the single crossing property in (β,s). In particular, for $\beta_2 \geq \beta_1$, we show that $\mathbb{T}R(\beta_2,s) \geq \mathbb{T}R(\beta_1,s)$ for any s. Denoting by b_i some element of $\arg\max_{b\in B_s}\{\pi(\beta_i,b,s)+\delta^2 R(b,s)\}$, we get

$$\pi(\beta_2,b_2,s)+\delta^2 R(b_2,s) \quad \geq \quad \pi(\beta_2,b_1,s)+\delta^2 R(b_1,s)$$

[26]Notice that $R_\beta(s,s) = R_\beta(s)$.

and so, for $\mathbb{T}R(\beta_2, s) \geq \mathbb{T}R(\beta_1, s)$ it is sufficient that

$$\pi(\beta_2, b_1, s) + \delta^2 R(b_1, s) \quad \geq \quad \pi(\beta_1, b_1, s) + \delta^2 R(b_1, s) \tag{42}$$

which for $P(b_1) > c(s)$ and $\beta_1 \geq b_1$ is equivalent to

$$\begin{aligned}(\beta_2 - b_1)(P(b_1) - c(s)) + \delta^2 R(b_1, s) \quad &\geq \quad (\beta_1 - b_1)(P(b_1) - c(s)) + \delta^2 R(b_1, s) \iff \\ (\beta_2 - \beta_1)(P(b_{21}) - c(s)) \quad &\geq \quad 0,\end{aligned}$$

which is implied by $\beta_2 \geq \beta_1$. When $P(b_1) \leq c(s)$ or $\beta_1 < b_1$, the inequality (42) is trivially satisfied.

Since the single crossing property is preserved under the pointwise limit, the solution R to the Bellman equation (11) satisfies the single crossing property in (b, s). Since $\frac{\partial}{\partial s}\pi(\beta, b, s) = -(\beta - P(\max\{\beta, b\})c'(s)$ is increasing in b, π has single crossing property in (b, s) and so function $\pi + \delta^2 R$ has single crossing property in (b, s). Since $\frac{\partial}{\partial \beta}\pi(\beta, b, s) = P(b) - c(s)$ is increasing in b for $b < \beta$ and is constant for $b \geq \beta$, $\pi + \delta^2 R$ has single crossing property in (b, β). Combined with the fact that set B_s is ascending, this implies, by Theorem 4 in Milgrom and Shannon (1994), that $\arg\max_{b \in B_s} \{\pi(\beta, b, s) + \delta^2 R(b, s)\}$ is ascending, and so t is non-decreasing in β and s. □

Lemma 16. *Consider the function π^i obtained on Step i of the iterative algorithm and let $\Pi^i(s)$ be the expected profit of type s that screens optimally facing the willingness to pay π^i. For all b, $\pi^i(b) \geq c(\underline{\pi}(b)) + (1 - \delta^2)\xi$, and for all $s \in [0, 1 - \eta]$, $\Pi^i(s) > C(\eta, \delta) > 0$ with $C(\eta, \delta) \sim (1 - \delta)^2$.*

Proof. For any type b of the buyer,

$$\begin{aligned}\pi^i(b) \quad &= \quad (1 - \delta^2)v(b) + \delta^2 \pi^{i-1}(\tau^i(\underline{\pi}(b))) \\ &\geq \quad (1 - \delta^2)v(b) + \delta^2 c(\underline{\pi}(b)) \\ &\geq \quad c(\underline{\pi}(b)) + (1 - \delta^2)\xi, \end{aligned} \tag{43}$$

where the first inequality follows from the fact that the seller gets non-negative profit when she best-replies to π^{i-1}, the second inequality follows from $v(b) - c(\underline{\pi}(b)) \geq \xi$.

Consider a type $s \in [0, 1 - \eta]$ and suppose she makes the offer $c(s) + (1 - \delta^2)\frac{\xi}{2}$. Consider $b \in B_s$ such that $c(\underline{\pi}(b)) + (1 - \delta^2)\xi \geq c(s) + (1 - \delta^2)\xi$. By (43), such types accept the offer $c(s) + (1 - \delta^2)\frac{\xi}{2}$. Since the derivative of c is bounded above by ℓ, the mass of such types is at least $\min\{2\eta, (1 - \delta^2)\frac{\xi}{2\ell}\}$. Therefore, type s is guaranteed to get at least $\min\{2\eta, (1 - \delta^2)\frac{\xi}{2\ell}\}(1 - \delta^2)\frac{\xi}{2\ell} = C(\eta, \delta)$ by offering $c(s) + (1 - \delta^2)\frac{\xi}{2}$. This minimal profit is equal to $(1 - \delta^2)^2 \frac{\xi^2}{4\ell}$ for δ close to one and, hence, $C(\eta, \delta) \sim (1 - \delta)^2$. □

Lemma 17. *There exists $c(\eta, \delta) > 0$ such that for all $s \in [0, \eta]$, $\overline{\pi}(s) - t(s) > c(\eta, \delta)$ and $c(\eta, \delta) \sim (1 - \delta)^3$ as δ goes to one.*

Proof. Make the change of variables $x = \overline{\pi}(s) - t(s)$ in the problem (11). Then $\Pi^i(s) = x(\pi^i(\overline{\pi}(s) - x) - c(s)) + \delta^2 \Pi^i(\overline{\pi}(s) - x, s) \leq x(\pi^i(\overline{\pi}(s) - x) - c(s)) + \delta^2(\Pi^i(s) + \ell_R x)$ where the inequality follows from the Lipschitz-continuity of $\Pi^i(s)$ (by Lemma 14). Therefore, $x \geq \frac{\Pi^i(s)(1-\delta^2)}{\pi^i(\overline{\pi}(s))-c(s)+\delta^2\ell_R} \geq \frac{C(\eta,\delta)(1-\delta^2)}{\Sigma+\ell_R}$ where we used the lower bound on $R(s)$ from Lemma 16. Letting $c(\eta,\delta) \equiv \frac{C(\eta,\delta)(1-\delta^2)}{\Sigma+\ell_R}$ gives the desired conclusion. □

Lemma 18. *On each step of the iterative algorithm, function $\pi^i(b)$ is right-continuous and strictly increasing.*

Proof. The proof is by induction on the step of the algorithm. For $i = 0$, the strict monotonicity of $\pi^0(b)$ follows from the strict monotonicity of P^0 and v, and the fact that $P^0(\eta) \leq v(\eta)$. The right-continuity of $\pi^0(b)$ follows from the right-continuity of P^0 and the continuity of v. Suppose by the inductive hypothesis that $\pi^{i-1}(b)$ is strictly increasing and right-continuous. For $b \in (b_i, 1]$, $\pi^i(b) = v(b)$ is strictly increasing and continuous. For $b \in [0, b_i]$, $\pi^i(b) = (1-\delta^2)v(b) + \delta^2 \hat{\pi}^{i-1}(\tau^i(\underline{\pi}(b)))$ is a convex combination of a strictly increasing function v and $\hat{\pi}^{i-1}(\tau^i(\underline{\pi}(b)))$. Function $\hat{\pi}^{i-1}(\tau^i(\underline{\pi}(b)))$ is increasing, as $\hat{\pi}^{i-1}$ is strictly increasing by the inductive hypothesis and $\tau^i(\underline{\pi}(b))$ is non-decreasing by Lemma 15. Therefore, $\pi^i(b)$ is strictly increasing on $[0, b_i)$. Moreover, $\pi^i(b_i) \leq v(b_i)$, which completes the proof of the strict monotonicity of $\pi^i(b)$. Function π^i is right-continuous, as it is a convex combination of right-continuous functions on $[0, b_i]$ and equal to continuous v on $(b_i, 1]$. □

Lemma 19. *Suppose $P(b)$ and $t(\beta, s)$ satisfy equations (2) and (11). Then for δ sufficiently close to one, in the punishing equilibrium strategies $P(b)$ and $t(\beta, s)$ are optimal on-path .*

Proof. From the design of the algorithm the screening strategy $t(\beta, s)$ is optimal for the seller who faces the willingness to pay P. We next show that the buyer does not have incentives to deviate either from the acceptance strategy P or from pooling on the price offer $\underline{y}(0, 0)$.

If the highest remaining type of the buyer exceeds b, then type b interprets the previous seller's offers as the seller's deviations and expects the seller to restart the screening. From equation (2), it follows that any offer above $P(b)$ would be rejected by buyer b. To complete the verification of optimality of the threshold strategy, we show that prices below $P(b)$ are accepted by buyer b.

Suppose to contradiction that the seller makes price offer p which is accepted by type b', but rejected by type $b > b'$. First, observe that if $b \leq \bar{\beta}$, then both types b and b' put probability one on type 0 of the seller, and the result follows from the single-crossing property of the payoffs. Next, suppose that $b' > \eta$. Let $\beta = \inf\{b : P(b) \geq p\}$. If the buyer rejects price offer p, then the highest type of the buyer remaining in the game is β. Each type s of the seller uses screening policy $t(\beta, s)$ after the rejection. Then for all $k \in \mathbb{N}$,

$$v(b') - p \geq \delta^{2k}\left(v(b') - P(t^{(k)}(\beta, \underline{\pi}(b')))\right) \quad (44)$$

$$v(b) - p < \delta^{2K}\left(v(b) - P(t^{(K)}(\beta, \underline{\pi}(b)))\right) \quad (45)$$

for some K, where $t^{(k)}(\beta, s)$ denotes the threshold type of the buyer to who type s of the seller sells in round k of the screening that started from type β being the highest type. That is, type b' accepts offer p, and buyer type b rejects such offer and expects to accept price offer $P(t^{(K)}(\beta, \underline{\pi}(b)))$ from type $\underline{\pi}(b)$ of the seller. Subtracting inequality (44) (with $k = K$) from (45), we get after rearranging terms

$$\left(1 - \delta^{2K}\right)(v(b) - v(b')) < -\delta^{2K}\left(P(t^{(K)}(\beta, \underline{\pi}(b))) - P(t^{(K)}(\beta, \underline{\pi}(b')))\right). \quad (46)$$

The left-hand side of (46) is greater than zero, as $b > b'$. By Lemma 15, $t^{(K)}(\beta, \underline{\pi}(b)) \geq t^{(K)}(\beta, \underline{\pi}(b'))$, and moreover, P is increasing. Hence, the right-hand side of (46) is less than zero, which gives a contradiction. If $b' \leq \bar{\beta} < b$, then the only difference with the previous case is that the screening of type 0 of the seller ends with the acceptance of $\underline{y}(0, 0)$. Therefore,

$$\begin{aligned} v(b') - p &\geq \delta\left(v(b') - \underline{y}(0,0)\right) \\ &\geq \delta^{2K}\left(v(b') - \underline{y}(0,0)\right) \end{aligned}$$

Combining this inequality with the same argument as before we get the contradiction.

Finally, we show that all types of the buyer prefer to pool on $\underline{y}(0, 0)$ for sufficiently large δ. By Theorem 1 any type $b > \eta$ of the buyer expects to get the good in the next round at price uniformly close to $P^*(b)$.[27] By Lemma 7, if such type deviates he trades with the seller at price close to $y(\underline{\pi}(b), b) > P^*(b)$ when δ is sufficiently large. Hence, such a deviation is not profitable. □

Proof of Theorem 5. By Lemma 17, the iterative algorithm converges in a finite number of steps. By Lemma 19, no deviations are profitable. □

Proof of Lemma 12

Proof. Observe that if x_k and y_k are given for $k \geq k_0$, then by (35), we can construct x_k and y_k for $k < k_0$. The following claim shows that it is sufficient to construct x_k and y_k that are positive starting from some k_0.

[27] Notice that the proof of Theorem 1 does not rely on the existence of the punishing equilibrium and simply analyzes the equilibrium path of the punishing equilibrium.

Claim 3. *If trajectories x_k and y_k satisfying* (35) *are positive starting from some k_0, then x_k and y_k are positive for all $k \in \mathbb{N}$.*

Proof. By rearranging terms in the first equation of (35), $x_k = \frac{x_{k+1}+\alpha^B(y_{k+1})y_{k+1}}{1-\alpha^B(y_{k+1})}$. Observe that $\alpha^B(y) \in (0,1)$ for $y > 0$ and so x_k is positive whenever x_{k+1} and y_{k+1} are positive. Analogously, it can be shown from the second equation of (35) that y_k is positive whenever x_{k+1} and y_{k+1} are positive. *Q.E.D.*

Claim 4. *For given x_{k_0} and y_{k_0}, there is $K(x_{k_0}, y_{k_0})$ such that $k_0 \leq K(x_{k_0}, y_{k_0})$.*

Proof. First, observe that x_k and y_k are decreasing whenever they are positive. Indeed, for all $k \in \mathbb{N}$, we have $x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) > 0$ and similarly, $y_{k-1} - y_k > 0$. Next, from (35), for all $k \leq k_0$,

$$x_{k-1} - x_k = \alpha^B(y_k)(x_{k-1} + y_k) \geq \alpha^B(y_{k_0})(x_{k_0} + y_{k_0}) > c_1 \tag{47}$$

for some $c_1 > 0$ where we made use of the fact that $\alpha^B(y)$ is increasing and x_k and y_k are decreasing sequences. Suppose for any $K \in \mathbb{N}$, we could construct sequences $x_k(K)$ and $y_k(K)$ such that $x_K(K) = x_{k_0}$ and $y_K(K) = y_{k_0}$. From (47), for K sufficiently large, $b_\infty + x_0(K) > s_\infty - y_0(K) + \eta$, which contradicts (35). *Q.E.D.*

Let $V^B \equiv v(b_\infty) - q^S$, $V^S \equiv q^B - c(s_\infty)$ and $\Delta P \equiv q^S - q^B$. The following claim gives rise to the Taylor expansion of $\alpha^B(y)$ and $\alpha^S(x)$.

Claim 5. *There exists $\delta_1 \in (0,1)$ and $\varepsilon_1 > 0$ such that for all $\delta \in (\delta_1, 1)$ and all $x \in (0, \varepsilon_1)$,$y \in (0, \varepsilon_1)$,*

$$\alpha^B(y) \equiv \alpha_B - \phi_B \sum_{l=1}^{\infty} \gamma_l^B y^l, \tag{48}$$

$$\alpha^S(x) \equiv \alpha_S - \phi_S \sum_{l=1}^{\infty} \gamma_l^S x^l, \tag{49}$$

where

$$\alpha_B \equiv \frac{(1-\delta^2)V^B}{\delta(\Delta P + (1-\delta)V^B)}, \gamma_B \equiv -\frac{1-\delta}{\Delta P + (1-\delta)V^B} < 0, \phi_B \equiv \frac{(1+\delta)\Delta P}{\delta(\Delta P + (1-\delta)V^B)} > 0,$$

$$\alpha_S \equiv \frac{(1-\delta^2)V^S}{\delta(\Delta P + (1-\delta)V^S)}, \gamma_S \equiv -\frac{1-\delta}{\Delta P + (1-\delta)V^S} < 0, \phi_S \equiv \frac{(1+\delta)\Delta P}{\delta(\Delta P + (1-\delta)V^S)} > 0,$$

$$\gamma_l^B \equiv \sum_{j=1}^{l} \gamma_B^j \left(\sum_{l_1+\cdots+l_j=l} \frac{d^{l_1}c(s_\infty)/ds^{l_1}}{l_1!} \cdots \frac{d^{l_j}c(s_\infty)/ds^{l_j}}{l_j!} \right),$$

$$\gamma_l^S \equiv \sum_{z=1}^{l} \gamma_S^z \left(\sum_{l_1+\cdots+l_z=l} \frac{d^{l_1}v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z}v(b_\infty)/db^{l_z}}{l_z!} \right),$$

and $\gamma_l^S \leq |\gamma_S D|(1+|\gamma_S D|)^{l-1}$, $\gamma_l^B \leq |\gamma_B D|(1+|\gamma_B D|)^{l-1}$.

Proof. As $\delta \to 1$, γ_S and γ_B converge to zero and so, for δ sufficiently close to one, $|\gamma_S(v(1)-v(0))| < 1$ and $|\gamma_B(c(1)-c(0))| < 1$. Expanding $\alpha^S(x)$ into the Taylor series, results in

$$\alpha^S(x) = \alpha_S - \phi_S \sum_{z=1}^{\infty} \gamma_S^z (v(b_\infty + x) - v(b_\infty))^z.$$

Since v is a smooth function, expanding it into the Taylor series around b_∞ results in $v(b_\infty + x) - v(b_\infty) = \sum\limits_{l=1}^{\infty} \frac{d^l v(b_\infty)}{db^l}\frac{x^l}{l!}$. By the regularity of v, all derivatives $\frac{d^l v(b)/db^l}{l!}$, $l \in \mathbb{N}$ are bounded by D for some $D > 1$. Therefore, the Taylor expansion of v around b_∞ is an absolute convergent series, and by the Merten's theorem the z's power of it equals

$$(v(b_\infty + x) - v(b_\infty))^z = \sum_{l=z}^{\infty} x^l \left(\sum_{l_1+\dots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right),$$

and so,

$$\alpha^S(x) = \alpha_S - \phi_S \sum_{z=1}^{\infty} \gamma_S^z \sum_{l=z}^{\infty} x^l \left(\sum_{l_1+\dots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right). \tag{50}$$

Observe that

$$\sum_{z=1}^{\infty} \left| \gamma_S^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\dots+l_z=l} \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \leq$$

$$\sum_{z=1}^{\infty} |\gamma_S|^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\dots+l_z=l} \left| \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \leq$$

$$\sum_{z=1}^{\infty} |\gamma_S|^z \sum_{l=z}^{\infty} x^l \sum_{l_1+\dots+l_z=l} D^z = \sum_{z=1}^{\infty} |\gamma_S|^z D^z \sum_{l=z}^{\infty} x^l \binom{l-1}{z-1} = \sum_{z=1}^{\infty} \frac{(|\gamma_S| D x)^z}{(1-x)^z} < \infty$$

where the first inequality follows from the triangle inequality, the second inequality follows from the regularity of v and the fact that $(l_1 + \ldots + l_z)! \geq l_1! \cdot \ldots \cdot l_z!$, the first equality follows from the fact that a number of compositions of l into exactly z parts is equal to $\binom{l-1}{z-1}$, the second equality results by summing over l, and the resulting series is convergent for x sufficiently small (so that $x < (1 + |\gamma_S|D)^{-1}$). Therefore, the series in (50) is absolutely convergent, and by the Fubini's theorem, we could exchange the order of summation in (50) to get expression (48). We have the following upper bound on the absolute values of coefficients γ_l^S

$$|\gamma_l^S| \leq \sum_{z=1}^{l} |\gamma_S|^z \left(\sum_{l_1+\cdots+l_z=l} \left| \frac{d^{l_1} v(b_\infty)/db^{l_1}}{l_1!} \cdots \frac{d^{l_z} v(b_\infty)/db^{l_z}}{l_z!} \right| \right) \leq \sum_{z=1}^{l} |\gamma_S D|^z \binom{l-1}{z-1} = |\gamma_S D|(1+|\gamma_S D|)^{l-1} \tag{51}$$

where the first inequality comes about via the triangle inequality, the second inequality follows from the regularity of v, and the equality is obtained by algebraic manipulation. The derivation of the corresponding expression for $\alpha^S(y)$ is analogous. *Q.E.D.*

System (35) has steady states $(z, -z), z \in \mathbb{R}$. By the specification of the problem we am interested only in steady state $(0, 0)$. Around this steady state the linearized system can be written in the matrix form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 - \alpha_B + \alpha_S\alpha_B & -\alpha_B(1-\alpha_S) \\ -\alpha_S & 1-\alpha_S \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

The matrix has eigenvalues 1 and $\lambda \equiv (1-\alpha_B)(1-\alpha_S)$. Since one of eigenvalues is equal to 1, the steady state is unstable, and we cannot conclude that in the neighborhood of the steady state the non-linear system will converge to the steady state. Therefore, we find a particular trajectory that satisfies desired properties.

We conjecture that there exist μ_i^x and μ_i^y such that

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sum_{i=1}^{\infty} \lambda^{ik} \begin{pmatrix} \lambda^{i/2}\mu_i^x \\ \mu_i^y \end{pmatrix}, \tag{52}$$

which is the required solution and for all $i \in \mathbb{N}$,

$$|\mu_i^x| \leq u_\delta M^i \text{ and } |\mu_i^y| \leq u_\delta M^i \tag{53}$$

for some positive M and u_δ such that

$$M < 1 < \frac{1}{\lambda(1 + u_\delta(1 + \max\{|\gamma_S|, |\gamma_B|\}D))}. \tag{54}$$

Given the accuracy of this conjecture, we next derive expressions for coefficients μ_i^x and μ_i^y, and then verify that for δ sufficiently close to one, upper bounds on absolute values of coefficients will hold. Series (52) defining (x_k, y_k) are absolutely convergent, as they are dominated by the absolutely convergent series $u_\delta \sum_{i=1}^{\infty} \lambda^{ik} M^i$.

Plugging the solution (52) into system (35), we get

$$\begin{cases} \sum\limits_{i=1}^{\infty} \lambda^{ik}(\mu_i^x - \mu_i^x\lambda^i - \alpha_B(\mu_i^x + \mu_i^y\lambda^{i/2})) = -\phi_B \left(\sum\limits_{l=1}^{\infty} \gamma_l^B \left(\sum\limits_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l \right) \left(\sum\limits_{i=1}^{\infty} \lambda^{ik}(\mu_i^x + \mu_i^y \lambda^{i/2}) \right), \\ \sum\limits_{i=1}^{\infty} \lambda^{ik}(\mu_i^y - \mu_i^y\lambda^i - \alpha_S(\mu_i^x\lambda^{i/2} + \mu_i^y)) = -\phi_S \left(\sum\limits_{l=1}^{\infty} \gamma_l^S \left(\sum\limits_{i=1}^{\infty} \mu_i^x \lambda^{ik} \right)^l \right) \left(\sum\limits_{i=1}^{\infty} \lambda^{i(k+1/2)}(\mu_i^x\lambda^{i/2} + \mu_i^y) \right). \end{cases} \tag{55}$$

Consider the first equation in system (55). By the Merten's Theorem, $\left(\sum\limits_{i=1}^{\infty} \mu_i^y \lambda^{i(k+1)} \right)^l = \sum\limits_{i=l}^{\infty} \sum\limits_{i_1+\dots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \lambda^{i(k+1)}$ and

$$\sum_{l=1}^{\infty} \gamma_l^B \left(\sum_{i=1}^{\infty} \mu_i^y \lambda^{ik} \right)^l = \sum_{l=1}^{\infty} \gamma_l^B \sum_{i=l}^{\infty} \sum_{i_1+\dots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \lambda^{i(k+1)}. \tag{56}$$

The series in (56) is absolutely convergent by

$$\sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \left| \lambda^{i(k+1)} \gamma_l^B \sum_{i_1+\dots+i_l=i} \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \right| \leq$$

$$\sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \lambda^{i(k+1)} |\gamma_l^B| \sum_{i_1+\dots+i_l=i} \left| \mu_{i_1}^y \cdot \ldots \cdot \mu_{i_l}^y \right| \leq$$

$$\sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \lambda^{i(k+1)} |\gamma_l^B| \sum_{i_1+\dots+i_l=i} u_\delta^l M^i = \sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \lambda^{i(k+1)} |\gamma_l^B| u_\delta^l M^i \begin{pmatrix} i-1 \\ l-1 \end{pmatrix} \leq$$

$$|\gamma_B D| \sum_{l=1}^{\infty} \sum_{i=l}^{\infty} \lambda^{i(k+1)} (1+|\gamma_B D|)^{l-1} u_\delta^l M^i \begin{pmatrix} i-1 \\ l-1 \end{pmatrix} = \frac{|\gamma_B D|}{1+|\gamma_B D|} \sum_{l=1}^{\infty} (1+|\gamma_B D|)^l u_\delta^l \left(\frac{\lambda^{k+1} M}{1-\lambda^{k+1} M} \right)^l \leq$$

$$\frac{|\gamma_B D|}{1+|\gamma_B D|} \sum_{l=1}^{\infty} (1+|\gamma_B D|)^l u_\delta^l \left(\frac{\lambda M}{1-\lambda M} \right)^l,$$

where the first inequality arises via the triangle inequality, the second inequality follows from (53), the first equality arises from the fact that the number of compositions of i into exactly l parts is $\begin{pmatrix} i-1 \\ l-1 \end{pmatrix}$. The third inequality follows from (51), the forth inequality is by $\lambda^{k+1} < \lambda$, and the resulting series is convergent whenever $u_\delta(1+|\gamma_B D|)\frac{\lambda M}{1-\lambda M} < 1$, which holds by (54).

Therefore, by Fubini's Theorem, exchanging the order of summation in (56) results in

$$\sum_{l=1}^{\infty}\gamma_l^B\left(\sum_{i=1}^{\infty}\mu_i^y\lambda^{ik}\right)^l=\sum_{i=1}^{\infty}\lambda^{i(k+1)}\sum_{l=1}^{i}\sum_{i_1+\cdots+i_l=i}\gamma_l^B\mu_{i_1}^y\cdot\ldots\cdot\mu_{i_l}^y.$$

By the absolute convergence of both series on the right-hand side of (55), the product on the right-hand side is equal to the Cauchy product, and so we can rewrite system (55) as follows

$$\begin{cases}\sum_{i=1}^{\infty}\lambda^{ik}\left(\mu_i^x-\mu_i^x\lambda^i-\alpha_B(\mu_i^x+\mu_i^y\lambda^{i/2})+\phi_B\sum\limits_{j=1}^{i-1}(\mu_{i-j}^x\lambda^{j/2}+\mu_{i-j}^y\lambda^{i/2})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y\right)=0,\\ \sum_{i=1}^{\infty}\lambda^{ik}\left(\mu_i^y-\mu_i^y\lambda^i-\alpha_S(\mu_i^x\lambda^{i/2}+\mu_i^y)+\phi_S\sum\limits_{j=1}^{i-1}(\mu_{i-j}^x\lambda^{i/2}+\mu_{i-j}^y\lambda^{j/2})\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x\cdot\ldots\cdot\mu_{j_l}^x\right)=0.\end{cases}$$

Setting all coefficient at λ^{ik} equal to zero results in the system

$$\begin{cases}\mu_i^x-\mu_i^x\lambda^i-\alpha_B(\mu_i^x\lambda^{j/2}+\mu_i^y\lambda^{i/2})=-\phi_B\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{j/2}+\mu_{i-j}^y\lambda^{i/2})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y\right),\\ \mu_i^y-\mu_i^y\lambda^i-\alpha_S(\mu_i^x\lambda^{i/2}+\mu_i^y)=-\phi_S\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{i/2}+\mu_{i-j}^y\lambda^{j/2})\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x\cdot\ldots\cdot\mu_{j_l}^x\right).\end{cases}$$

Using notation $A_i\equiv\begin{pmatrix}1-\lambda^i-\alpha_B & -\alpha_B\lambda^{i/2}\\ -\alpha_S\lambda^{i/2} & 1-\lambda^i-\alpha_S\end{pmatrix}$, $\mu_i\equiv\begin{pmatrix}\mu_i^x\\ \mu_i^y\end{pmatrix}$, and

$$\varphi_i=\begin{pmatrix}\varphi_i^x\\ \varphi_i^y\end{pmatrix}\equiv\begin{pmatrix}-\phi_B\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{j/2}+\mu_{i-j}^y\lambda^{i/2})\sum\limits_{l=1}^{j}\gamma_l^B\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y\right)\\ -\phi_S\sum\limits_{j=1}^{i-1}\left((\mu_{i-j}^x\lambda^{i/2}+\mu_{i-j}^y\lambda^{j/2})\sum\limits_{l=1}^{j}\gamma_l^S\sum\limits_{j_1+\cdots+j_l=j}\mu_{j_1}^x\cdot\ldots\cdot\mu_{j_l}^x\right)\end{pmatrix},\tag{57}$$

and we can write the system in matrix form as $A_i\mu_i=\varphi_i$. Since $\det(A_i)=(1-\lambda^i)(\lambda-\lambda^i)>0$, for $i\geq 2$, matrix A_i is invertible, and we can solve for all μ_i (with the exception of $i=1$)

$$\mu_i=A_i^{-1}\varphi_i.\tag{58}$$

For $i=1$, the equations are linearly dependent and the relation between μ_1^x and μ_1^y is given by

$$\mu_1^x=\mu_1^y\frac{\alpha_B}{\alpha_S}(1-\alpha_S).\tag{59}$$

Equations (58) and (59) give the desired expressions for μ_i through the parameters of the model. The next claim verifies that bounds (53) and (54) indeed hold and so, my derivation is justified.

Claim 6. *For $M<1$, there exists $\hat{\delta}\in(0,1)$ such that for any $\delta\in(\hat{\delta},1)$ there exist positive u_δ and μ_1^y such that* (54) *holds, and for μ_i defined by (58) and* (59), *bounds* (53) *hold.*

Proof. The proof is by induction on i. Without loss of generality, we assume that

$$V^S \leq V^B \tag{60}$$

and so, $\alpha_S \leq \alpha_B, |\gamma_S| \geq |\gamma_B|, \phi_S \geq \phi_B$. Let $u_\delta \equiv \frac{u}{2}\min\{|\gamma_S|, |\gamma_B|\}$ where $u = \frac{1}{2}\min\{V^S, V^B\}$. Let us first show that for our choice of u_δ, $1 < \frac{1}{\lambda(1+u_\delta(1+\max\{|\gamma_S|,|\gamma_B|\}D))}$ for δ sufficiently close to one. To see this, observe that for δ sufficiently close to one, $\max\{|\gamma_B|, |\gamma_S|\}D < 1$ and so, $\frac{1}{\lambda(1+2u_\delta)} < \frac{1}{\lambda(1+u_\delta(1+\max\{|\gamma_S|,|\gamma_B|\}D))}$. Therefore, it is sufficient to show that $\lambda^{1/2}(1+2u_\delta) < 1$. Then

$$\lambda^{1/2}(1+2u_\delta) = ((1-\alpha_S)(1-\alpha_B))^{1/2}(1+u\min\{|\gamma_S|, |\gamma_B|\}) \leq (1-\alpha_S)(1+u|\gamma_S|).$$

Observe

$$(1-\alpha_S)(1+u|\gamma_S|) = \left(1 - \frac{(1-\delta^2)V^S}{\delta(\Delta P + (1-\delta)V^S)}\right)\left(1 + \frac{(1-\delta)u}{\Delta P + (1-\delta)V^S}\right),$$

and $\lambda^{1/2}(1+2u_\delta) < 1$ is equivalent to

$$\Delta P + (1-\delta)V^S + u(1-\delta) < \frac{\delta(\Delta P + (1-\delta)V^S)(\Delta P + (1-\delta)V^S)}{\Delta P\delta - (1-\delta)V^S},$$

or

$$u < (1+\delta)V^S \frac{\Delta P + (1-\delta)V^S}{\Delta P\delta - (1-\delta)V^S}. \tag{61}$$

As $\delta \to 1$, the right-hand side of (61) converges to $2V^S$. Since $u < V^S$, inequality (61) holds and so $(1-\alpha_S)(1+u|\gamma_S|) < 1$ for sufficiently large δ. Hence, we have proven that (54) holds.

To prove bounds (53), let μ_1^x and μ_1^y be defined as follows. If $\frac{\alpha_B}{\alpha_S}(1-\alpha_S) \leq 1$, then let $\mu_1^y = u_\delta M$ and $\mu_1^x = \mu_1^y \frac{\alpha_B(1-\alpha_s)}{\alpha_S} \leq \mu_1^y$, and otherwise let $\mu_1^x = u_\delta M$ and $\mu_1^y = \mu_1^x \frac{\alpha_S}{\alpha_B(1-\alpha_S)} \leq \mu_1^x$. By the definition, $|\mu_1^x|$ and $|\mu_i^y|$ are less than $u_\delta M$, which proves the base of induction.

Suppose that the statement is true for all $j < i$. We show that $|\mu_i^x| < u_\delta M^i$ and $|\mu_i^y| < u_\delta M^i$. We can find closed-form solution to system (58),

$$|\mu_i^x| = \frac{|(1-\lambda^i-\alpha_S)\varphi_i^x + \alpha_B\lambda^{i/2}\varphi_i^y|}{(1-\lambda^i)(\lambda-\lambda^i)} \leq \frac{4\max\{1-\lambda^i, \alpha_S, \alpha_B\}\cdot\max\{|\varphi_i^x|, |\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)}$$

and the same upper bound holds for $|\mu_i^y|$. It is sufficient to show that $\frac{4\max\{(1-\lambda^i),\alpha_S,\alpha_B\}\cdot\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(1-\lambda^i)(\lambda-\lambda^i)u_\delta M^i} < 1$.

Notice that $\frac{\alpha_S}{1-\lambda^i} < \frac{\alpha_S}{1-\lambda}$ for $i \geq 2$, and by l'Hospital rule $\lim_{\delta\to 1}\frac{\alpha_S}{1-\lambda} = \lim_{\delta\to 1}\frac{\alpha_S}{\alpha_S+\alpha_B-\alpha_S\alpha_B} = \frac{V^S}{V^S+V^B} \leq 1$. Hence, for sufficiently large δ and all $i \geq 2$, we have $\frac{\alpha_S}{1-\lambda^i} < 1$, and by an analogous argument, $\frac{\alpha_B}{1-\lambda^i} < 1$. Therefore, $\frac{4\max\{1-\lambda^i,\alpha_S,\alpha_B\}}{1-\lambda^i} < 5$ for sufficiently large δ and it remains to show that $\frac{\max\{|\varphi_i^x|,|\varphi_i^y|\}}{(\lambda-\lambda^i)u_\delta M^i} < \frac{1}{5}$ for sufficiently large δ.

We next show that $\frac{|\varphi_i^x|}{(\lambda-\lambda^i)u_\delta M} < \frac{1}{5}$ (by the symmetric argument $\frac{|\varphi_i^y|}{(\lambda-\lambda^i)u_\delta M} < \frac{1}{5}$). From (57) it follows

$$\frac{|\varphi_i^x|}{\phi_B} \leq \sum_{j=1}^{i-1}\lambda^{j/2}\sum_{l=1}^{j}|\gamma_l^B|\sum_{j_1+\cdots+j_l=j}|\mu_{i-j}^x\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y| + \lambda^{i/2}\sum_{j=1}^{i-1}\sum_{l=1}^{j}|\gamma_l^B|\sum_{j_1+\cdots+j_l=j}|\mu_{i-j}^y\mu_{j_1}^y\cdot\ldots\cdot\mu_{j_l}^y| \leq$$

$$\sum_{j=1}^{i-1}\lambda^{j/2}\sum_{l=1}^{j}|\gamma_l^B|\sum_{j_1+\cdots+j_l=j}u_\delta^{l+1}M^i + \lambda^{i/2}\sum_{j=1}^{i-1}\sum_{l=1}^{j}|\gamma_l^B|\sum_{j_1+\cdots+j_l=j}u_\delta^{l+1}M^i \leq$$

$$2u_\delta M^i\sum_{j=1}^{i-1}\lambda^{j/2}\sum_{l=1}^{j}|\gamma_l^B|u_\delta^l\binom{j-1}{l-1} \leq$$

$$2u_\delta M^i|\gamma_B D|\sum_{j=1}^{i-1}\lambda^{j/2}\sum_{l=1}^{j}u_\delta^l(1+|\gamma_B D|)^{l-1}\binom{j-1}{l-1} \leq$$

$$2u_\delta M^i|\gamma_B D|\sum_{j=1}^{i-1}\lambda^{j/2}u_\delta\left(1+u_\delta(1+|\gamma_B D|)\right)^{j-1} \leq$$

$$2u_\delta M^i|\gamma_B D|\sum_{j=1}^{i-1}\lambda^{j/2}u_\delta(1+2u_\delta)^{j-1} =$$

$$2u_\delta M^i|\gamma_B D|\frac{u_\delta\lambda^{1/2}(1-\lambda^{(i-1)/2}(1+2u_\delta)^{i-1})}{1-\lambda^{1/2}(1+2u_\delta))},$$

where the first inequality is due to the triangle inequality, the second inequality arises via the inductive hypothesis, the third inequality makes use of the fact that the number of compositions of j into exactly l parts is $\binom{j-1}{l-1}$ and that $\lambda^j > \lambda^i$ for $j < i$, the forth inequality uses a bound on $|\gamma_l^B|$, the fifth inequality exists by summing over l, the sixth inequality is by $|\gamma_B D| < 1$ for sufficiently large δ, the equality is the summation over j. It remains to show that

$$2\phi_B|\gamma_B D|\frac{u_\delta\lambda^{1/2}(1-\lambda^{(i-1)/2}(1+2u_\delta)^{i-1})}{(\lambda-\lambda^i)(1-\lambda^{1/2}(1+2u_\delta))} < \frac{1}{5}. \quad (62)$$

Since the denominator in (62) is positive, (62) is equivalent to

$$\lambda-\lambda^i-10\phi_B|\gamma_B D|\frac{u_\delta\lambda^{1/2}(1-\lambda^{(i-1)/2}(1+2u_\delta)^{i-1})}{1-\lambda^{1/2}(1+2u_\delta)} > 0. \quad (63)$$

The derivative of (63) with respect to i is equal to

$$\lambda^{i/2}\left(-\ln(\lambda)\lambda^{i/2}+10\ln(\lambda^{1/2}(1+2u_\delta))\phi_B|\gamma_B D|\frac{u_\delta(1+2u_\delta)^{i-1}}{1-\lambda^{1/2}(1+2u_\delta)}\right).$$

Multiplication by $\lambda^{i/2}$ does not affect the sign of the derivative so we focus on the term in brackets. The positive (first) term in brackets is decreasing in absolute value, while the negative (second) term is increasing in absolute value. Therefore, the minimum of expression (63) is either attained at $i=2$ or $i\to\infty$. For $i=2$, (63) is equal to

$$\lambda-\lambda^2-10\phi_B|\gamma_B D|u_\delta\lambda^{1/2}>0, \tag{64}$$

whenever $u_\delta<\frac{\lambda^{1/2}(1-\lambda)}{10\phi_B|\gamma_B D|}$. By (60), $\frac{\lambda^{1/2}(1-\lambda)}{10\phi_B|\gamma_B D|}=\frac{\lambda^{1/2}(1-(1-\alpha_B)(1-\alpha_S))}{10\phi_B|\gamma_B D|}\leq\frac{\lambda^{1/2}(1-(1-\alpha_B)^2)}{10\phi_B|\gamma_B D|}\leq\frac{\alpha_B}{\phi_B|\gamma_B|}\to V^B$. Since u_δ converges to zero as $\delta\to 1$, inequality (64) holds for δ close to one.

For $i=\infty$, (63) is equal to

$$\lambda\left(1-10D\frac{\phi_B|\gamma_B|}{\lambda^{1/2}}\frac{u_\delta}{1-\lambda^{1/2}(1+2u_\delta)}\right). \tag{65}$$

Observe that $\lim\limits_{\delta\to 1}\frac{u_\delta}{1-\lambda^{1/2}(1+2u_\delta)}=\frac{u}{V^S+V^B-2u}$. Since $|\gamma_B|\to 0,\lambda\to 1,\phi_B\to 2$ as $\delta\to 1$, we have that (65) is positive for sufficiently large δ. *Q.E.D.*

So far we have constructed the candidate trajectories (x_k,y_k) given by (52). First, notice that by making k sufficiently large the solution approaches zero and so, the Taylor expansion in Claim 5 is justified. Second, observe that $x_k=\sum_{i=1}^{\infty}\lambda^{ik}\lambda^{i/2}\mu_i^x=\lambda^{k+1/2}\left(\mu_1^x+\sum_{i=2}^{\infty}\lambda^{(i-1)k}\lambda^{i/2}\mu_i^x\right)$, and for sufficiently large k , the sign of x_k is determined by μ_1^x which we can choose to be positive. Analogously, since μ_1^y has the same sign as μ_1^x (by the definition), y_k is positive for sufficiently large k. By Claim 3, the constructed trajectory (x_k,y_k) is positive. □