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Asset Dissemination through Dealer Markets

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Abstract

In over-the-counter markets for assets such as bonds or securitizations, large volumes can be split into smaller pieces and gradually sold to several final investors with the intermediation of multiple dealers. This paper proposes a model to study this process, called "asset dissemination". A dealer buys several units of an asset from a customer, then sells some units to his customers and to a second dealer, who sells to his customers and a third dealer, and so on. The extent of dissemination is measured by the number of dealers involved and the total customer demand served. We show that asymmetric information on customer demand hinders both dimensions of dissemination. We also study how the quantity to disseminate and the dealers' funding costs impact dissemination and the prices and quantities in interdealer transactions.

Journal of Economic Literature Classification Number: C78, D85, G21, G23.
Keywords: intermediation chains, liquidity, OTC markets, dealer markets.

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1 Introduction

Many important assets, for instance bonds, are traded on over-the-counter (OTC) markets rather than on an exchange. A recent literature precisely documents the pattern of intermediation on these markets. In particular, a significant fraction of trading volume is due to interdealer transactions. Many such transactions belong to "intermediation chains", in which dealers pass a constant quantity to each other before it reaches a final investor. In some other transactions, a large initial quantity is split into smaller pieces that are sold to different ultimate buyers via several dealers, a phenomenon we call "asset dissemination through dealer markets".[1]

A growing literature has analyzed intermediation chains and their relation with trading costs for customers, but asset dissemination has received less attention.[2] We provide a theoretical framework in which dissemination occurs because each dealer has convex holding costs and may not face enough customer demand to absorb a large sale. We endogenize the number of dealers disseminating an asset, the inventories they keep, and the purchases made by their customers. We show that asymmetric information on the customer demand faced by each dealer hinders dissemination. We then derive predictions about the joint distribution of the number of dealers involved in disseminating an asset (which we call the "dissemination length"), the transactions they conduct, and the volumes and prices of these transactions, thus providing some theoretical guidance for empirical research on asset dissemination.

Our model starts with a customer who offers to sell a certain quantity of an asset (the "supply"), which, if accepted, will be disseminated to several buyers (the "demand") by a sequence of dealers. The dissemination length is endogenous and depends on the supply by the initial seller, the (stochastic) demand of other customers, and informational frictions

[1]The relative importance of both forms of intermediation varies across markets. For U.S. corporate bonds Friewald and Nagler (2019) report that 43% of the trading volume they analyze takes place in sequences involving at least two dealers selling to at least two clients. For registered securitizations Hollifield, Neklyudov, and Spatt (2017) (Table 3) show that about 27% of sales by customers give rise to a sequence involving more than one dealer, in which case on average 2.8 dealers disseminate the asset to 3.7 different clients. For municipal bonds Li and Schürhoff (2019) find that intermediation chains account for 82% of the trades that start with a customer selling to a dealer.

[2]Intermediation chains feature for instance in Glode and Opp (2016) and Hugonnier, Lester, and Weill (2019). Viswanathan and Wang (2004) and Uslu (2019) are two papers that model dissemination. We review these papers in detail at the end of this section.

in interdealer trades. Dissemination length can correlate either positively or negatively with other variables of interest such as transaction prices, depending on which type of shock drives these variations. On the supply side, our model predicts that higher quantities to disseminate lead to a higher dissemination length and larger trading volumes. The prices are not necessarily lower, but the "intermediation revenues", the discount offered by a dealer times the quantity sold, are larger. On the demand side, we show that asymmetric information about customer demand leads to both lower dissemination length and lower transaction prices.

More precisely, we consider a model in which a first dealer buys a certain quantity of an asset from a client. This dealer can sell part or all of this quantity to his customers, who are in limited number. In particular, selling to customers may not be enough to finance the dealer's purchase of the asset, in which case he can contact a second dealer, and make a take-it-or-leave-it offer specifying a price and a number of units.[3] If the second dealer accepts the offer, she can in turn sell to her customers and/or contact another dealer. Transactions continue until a dealer turns down the offer he receives, or does not make a new offer. At the end of the process, dealers have disseminated part of the assets to their customers, and they keep the rest as inventory. Importantly, all dealers can also finance a purchase by borrowing cash but face a "funding spread".

This game is a parsimonious way to model the trading and dissemination of a divisible asset, assuming several frictions. First, trades can only occur between dealers with a preexisting relationship, as is documented by the growing literature on OTC trading. Second, while the asset's value is commonly known, each dealer's customer demand is private information. Third, borrowing from external financiers is costly to dealers. This introduces a convexity in dealers' holding costs and gives each dealer an incentive to sell the units he bought to someone else. Hence, the motive for interdealer trades is to reach the (unknown) customer demand of other dealers to save on funding costs.[4]

[3] Thus, our model applies to markets in which interdealer transactions are mostly bilateral, which is still the case in many markets. On the corporate bond market for instance, Bessembinder, Spatt, and Venkataraman (2020) report that only 23% of trading volume in investment grade corporate bonds is facilitated electronically. Our model in its current form also does not allow for pre-arranged trades. According to Harris (2015) (Tables 9 and 25), about 68% of trades in the retail segment of the corporate bond market are pre-arranged trades in which the dealer does not take the risk of keeping the asset on inventory. This number falls to less than 8% for institutional-size trades.

[4] Neither risk-sharing nor information about the asset's value play a role in the model.

We build an equilibrium of this game in which a dealer's offer depends only on the difference between the amount he has to pay to the previous dealer and the amount he can collect by selling assets to his customers. We call this quantity the dealer's financing needs.

A dealer with higher financing needs is more eager to trade with another dealer to avoid costly borrowing. As a result, he makes offers that are more "generous", i.e., accepted by dealers with a lower customer demand, which leads to the involvement of more dealers in the chain of transactions. In order to increase the probability of acceptance, the dealer has to offer higher rents to the next dealer, which is done by increasing intermediation revenues.[5] This explains why the model generates a positive correlation between dissemination length and intermediation revenues. Interestingly, because we have a multi-unit environment a dealer can offer higher intermediation revenues through a combination of higher prices and higher volumes, rather than lower prices.

If all dealers were perfectly informed about how much customer demand all the other dealers face, each of them would make a take-it-or-leave-it offer leaving no surplus to the next dealer. As a result all transactions would involve just as many dealers as is necessary to entirely disseminate the asset to customers, and all interdealer prices would be equal to the asset's fundamental value. Instead, when each dealer makes an offer under asymmetric information, he needs to leave some informational rent to his counterparty, so that the asset trades at prices below the fundamental value. Moreover, in order to reduce this rent, each dealer chooses an offer with a positive probability of rejection, so that the dissemination process can stop inefficiently early. This explains why asymmetric information leads to lower asset prices, lower dissemination length, and lower customer purchases.

Our framework allows us to derive a number of testable implications on the impact of the funding spread. Higher funding spreads make it more valuable for dealers to find a buyer for the asset rather than finance it via borrowing. As a result, a higher funding spread leads to offers with lower prices and larger volumes. Moreover, the impact is not the same for all dealers. The initial dealer keeps a smaller inventory of the asset and obtains a smaller expected profit, while subsequent dealers keep larger inventories and obtain larger expected

[5]In our model, asset dissemination is thus determined by a classical trade-off between rent extraction and efficiency. The problem is compounded at each level in the sequence of transactions, which is reminiscent of the literature on double marginalization (going back to Spengler (1950)).

profits. To our knowledge, these are new testable predictions.

In addition to delivering new insights regarding the joint distribution of dissemination length, prices, and volumes, our model contributes to the literature on OTC markets by introducing and solving a tractable framework that combines: (i) intermediation by an a priori unbounded number of dealers; (ii) a non-stationary environment, in which an asset is gradually disseminated over time; (iii) a divisible asset and convex holding costs, so that traded volumes are non-trivial; (iv) offers made under asymmetric information about dealers' endowments (but symmetric information about the asset).

To solve the model we exploit the fact that, even though the environment is not stationary, two dealers at different positions in a sequence of transactions who have the same endowments and receive the same offers actually face the exact same strategic situation. We thus focus on equilibria in which their behavior is also the same. We then build a particularly tractable equilibrium in which a dealer's offer is characterized by a "target", the dealer type indifferent between accepting and rejecting the offer, and his equilibrium payoff is characterized by a function called the "collected demand". Both the target and the collected demand depend only on the dealer's financing needs. The target of a given dealer has to be optimal given the collected demand of the dealer who receives the offer. We deduce that the collected demand has to satisfy a particular functional equation, the solution to which characterizes the equilibrium behavior of all players in the model.

In Section 6 we illustrate the usefulness and flexibility of our framework by considering several extensions. We first show variants of the model that account for more complicated forms of intermediation that have been documented in the literature, such as one dealer selling to multiple dealers, or dealers who do not have customers themselves. We also analyze a case in which dealers face inventory costs instead of funding costs. Finally, we consider an extension in which dealers obtain financing by using their assets as collateral in repo transactions. In this setting the dissemination length can be interpreted as a measure of collateral rehypothecation, the extent of which has attracted the attention of policymakers.

Related literature. The focus of our paper is on the dissemination of an asset when trading is decentralized and constrained by pre-existing relationships and asymmetric information. Two approaches have been followed to model decentralized trading in a tractable way, namely networks and random matching with search. In either case, a model aiming at studying dissemination by dealers needs two ingredients: (i) an endogenous number of intermediaries between an initial seller and final buyers; (ii) a divisible asset and a non-trivial quantity choice by each dealer. To our knowledge this combination has been studied by only few papers in the literature, none of which studies dissemination.

Network models assume that transactions are only possible between pairs of linked agents, so that there is some stability in who trades with whom, as is observed empirically. Most papers in this literature consider the trading of a single indivisible object, so that dissemination is not a relevant question. Intermediation then means that a trader buys and resells the object, and the main question is whether the object is ultimately owned by the buyer with the highest valuation (see, e.g., Condorelli, Galeotti, and Renou (2016), Manea (2018), and Gofman (2014)). An important exception is Malamud and Rostek (2017), who develop a general model of decentralized trading with divisible assets but mainly study the implications on liquidity, not on dissemination.

The closest paper using preexisting relationships is Viswanathan and Wang (2004), Sections IV and VI, which analyzes a sequential inter-dealer trading mechanism. The asset is divisible and risky and dealers are risk-averse. The asset is then disseminated among dealers to share the risk. The successive prices reflect risk premia, which decrease along the transactions as the volume decreases along the sequence. Crucially, the number of traders who can hold the asset is predetermined, so that dissemination is not endogenous. Rather, the number of dealers involved is fixed, and the model studies the implications of risk-sharing among dealers for asset prices.

Another very related paper is Glode and Opp (2016), which also studies a sequential screening game between intermediaries. They show that efficiency is improved in a market with adverse selection by introducing sufficiently many moderately informed intermediaries between sellers and buyers.[6] The offers made in equilibrium can be rejected with a positive probability, so that who holds the asset is endogenous. However, the utility of each agent is linear in the quantity held, so that it is suboptimal to split the initial quantity.[7]

[6] Glode, Opp, and Zhang (2019) study the informational conditions under which such chains implement efficient trades.

[7] Back, Liu, and Teguia (2020) study an interesting environment in which the action of a dealer also has

Models of random matching with search, initiated by Rubinstein and Wolinsky (1987), are the foundation of recent models of OTC markets starting with Duffie, Garleanu, and Pedersen (2005). Traders meet randomly in pairs, bargain and either reach an agreement, or fail to do so and search for another partner. Several papers in this literature build models that generate intermediation chains with an indivisible asset, thus without dissemination. In particular, Hugonnier, Lester, and Weill (2019), Shen, Wei, and Yan (2020), and Sambalaibat (2018) derive results on the correlation between chain length and mark-ups charged to customers. We abstract from some important features of dealer markets studied in these papers,[8] notably the core-periphery structure analyzed in Sambalaibat (2018), in order to have a tractable model with a divisible asset.

Existing search models with non-linear holding costs do not focus on asset dissemination. Uslu (2019) introduces a convex holding cost in a generalized version of Duffie, Garleanu, and Pedersen (2007),[9] giving traders an incentive to share the asset in a non-trivial way. The paper focuses on the relation between asset prices and centrality rather than on dissemination length. Cujean and Praz (2015) develop a search model with asymmetric information on inventories and focus on the impact of making information more transparent. Afonso and Lagos (2015) assume frictionless Nash bargaining so that the traders who meet choose quantities that equalize their inventories, there is thus no strategic component in the determination of quantities.

A different approach is followed by Lyons (1997), who develops a model in which dealers post quotes on a centralized platform. The model generates "hot-potato trading", which is quite different from dissemination as each dealers pass inventory imbalances to each other "independently of whether they offset the imbalance of the receiving dealer".

a signaling component. The asset is indivisible and the dealer only intermediates between two customers.

[8]As well as Wright and Wong (2014), Neklyudov (2019), and Farboodi, Jarosch, and Shimer (2020).

[9]See also Lagos and Rocheteau (2009) and Atkeson, Eisfeldt, and Weill (2015).

2 The model

2.1 The game

Environment. We consider an asset for which there is no centralized market and trading takes place bilaterally between two dealers or between a dealer and a customer. The asset pays a sure return $\rho > 0$ per unit. There is an exogenous sequence $\{D_n\}_{n\geq 1}$ of risk-neutral dealers, whose customers have a certain demand for the asset: D_n's customers are willing to buy up to v_n^C units at price ρ. In case the dealer supplies the maximum quantity v_n^C, she receives $\omega_n = \rho v_n^C$. We call ω_n the customer demand of D_n.

The customer demands of all dealers are identically and independently distributed according to a distribution G over $\mathbb{R}^+$: $G(\omega) = \Pr(\tilde{\omega} \leq \omega)$, with $\mathbb{E}(\tilde{\omega})$ finite.[10] It is also convenient to define H by $H(\omega) = \Pr(\tilde{\omega} \geq \omega)$.

The game is a succession of take-it-or-leave-it offers along the sequence of dealers. The game starts with dealer D_1, who has received an exogenous take-it-or-leave-it offer (p_0, v_0) (e.g., from a client selling the asset), which gives her the opportunity to buy v_0 units of the asset at a unit price $p_0 \leq \rho$. If D_1 accepts (p_0, v_0), she can make a take-it-or-leave-it offer (p_1, v_1) to D_2, who can make an offer (p_2, v_2) to D_3. At each further step, a dealer D_n who accepts an offer from D_{n-1} can then make a new offer (p_n, v_n) to D_{n+1}. Each dealer is available to trade with a probability $q \in (0, 1)$, and the customer demand ω_n is privately known by D_n. Thus, when making an offer, D_n does not know whether D_{n+1} is available, nor ω_{n+1}. Importantly, D_n has to decide whether to accept an offer before making a new offer to D_{n+1}.

Trading takes place until a dealer is inactive, turns down the received offer, or makes no new offer. When this happens, interdealer transactions are settled. The sellers transfer the units of the asset to their customers and, if applicable, to other dealers, in exchange for cash. As we will detail below, the cash received may not be enough to pay for the units purchased. We assume a dealer can always borrow cash at a net interest rate $r > 0$ from external financiers. A surplus of cash does not yield any interest so the value r can be interpreted as a *funding spread* paid by the dealer.

[10] We will introduce additional regularity assumptions for some results.

The game is entirely parameterized by the initial offer (p_0, v_0), the value ρ of the asset, the funding spread r, the probability q, and the distribution G. These parameters are common knowledge to all players.

Payoffs. Let us describe the problem faced by a dealer D_n, who has received an offer (p_{n-1}, v_{n-1}) and whose customer demand is ω_n. We assume in this description that prices, here p_{n-1} and p_n, are not higher than ρ.[11] If D_n is inactive, or if D_n rejects the offer, then D_n's payoff is null. If D_n accepts the offer, she receives v_{n-1} units and has to pay $p_{n-1}v_{n-1}$. D_n can sell up to $\min(v_n^C, v_{n-1})$ units to her customers and receive $\rho \min(v_n^C, v_{n-1})$ units of cash in exchange. If this amount is smaller than $p_{n-1}v_{n-1}$, D_n needs to find additional cash. Since $p_{n-1} \leq \rho$, this occurs only if $v_{n-1} > v_n^C$, in which case D_n needs to find the following amount of cash y_n, called D_n's *financing needs*:

$$y_n = \max(p_{n-1}v_{n-1} - \omega_n, 0). \tag{1}$$

In order to cover her financing needs, D_n can sell some units to dealer D_{n+1}, by making a take-it-or-leave-it offer (p_n, v_n). She cannot sell more units than she bought, so that $v_n \leq v_{n-1}$. We call *feasible* such an offer. Choosing $v_n = 0$ is interpreted as making no offer.

If D_{n+1} accepts the offer (p_n, v_n), D_n transfers v_n units to D_{n+1} against a payment of $p_n v_n$ in cash, sells $\min(v_n^C, v_{n-1} - v_n)$ units to her customers and receives $\rho \min(v_n^C, v_{n-1} - v_n)$ in cash in exchange. If $p_n v_n$ does not cover the financing needs y_n, then D_n needs to borrow $y_n - p_n v_n$ at rate r, otherwise she has a positive cash position earning a null return. If instead D_{n+1} is inactive or rejects D_n's offer, then D_n needs to borrow y_n at rate r. To summarize, D_n's payoffs are:

$$0 \qquad \text{if } D_n \text{ rejects } (p_{n-1}, v_{n-1}) \tag{2}$$

$$(\rho - p_{n-1})v_{n-1} - r y_n \qquad \text{if } D_n \text{ accepts and } D_{n+1} \text{ rejects } (p_n, v_n) \tag{3}$$

$$(\rho - p_{n-1})v_{n-1} - (\rho - p_n)v_n - r\max(y_n - p_n v_n, 0) \qquad \text{if } D_n \text{ accepts and } D_{n+1} \text{ accepts } (p_n, v_n). \tag{4}$$

[11] This is true in the equilibria we consider. The Online Appendix discusses bubble equilibria with prices above ρ.

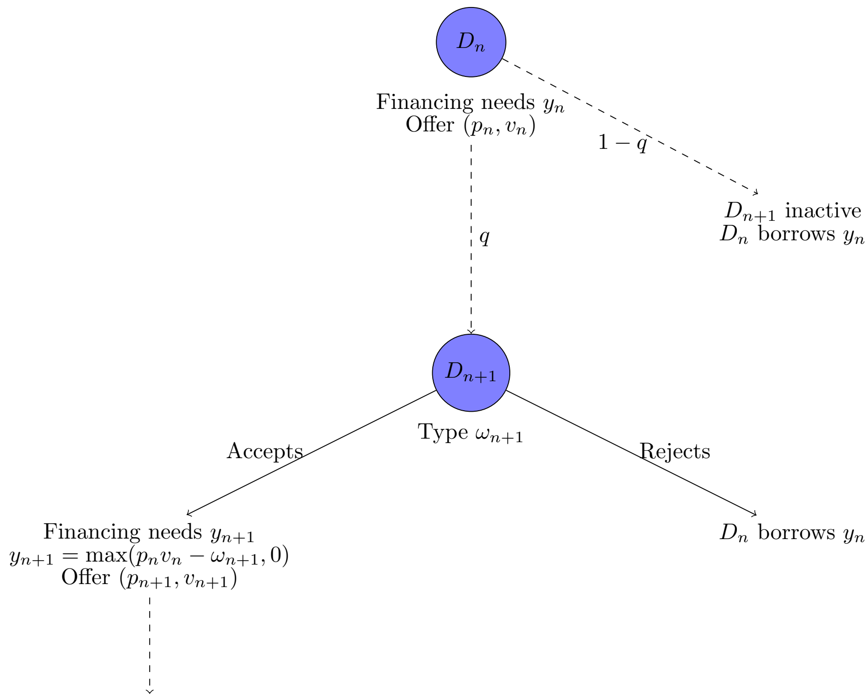

Figure 1 – Trading process.

The expression in (3) is called the *stand-alone profit*, which is obtained if D_n accepts the received offer and her offer is rejected, or she does not make an offer ($v_n = 0$). It is made of the *intermediation revenues* $(\rho - p_{n-1})v_{n-1}$ minus the financing costs. The game continues only if D_{n+1} accepts D_n's offer. In that case, D_{n+1} faces the same problem as described for D_n, and the payoffs are the same as above, with indices raised by one unit. The process thus stops as soon as a dealer is inactive, turns down the received offer, or makes no new offer. Fig. 1 summarizes the trading process.

Benchmark: complete information. As a benchmark, consider a variant of the model with complete information. Each dealer D_n is informed about which other dealers are available for trade, and about all customer demands. In particular, the number M of successive dealers who are active starting from D_1 is common knowledge (that is, D_{M+1} is the first inactive dealer), as well as the sequence $\{\omega_1, \omega_2 ... \omega_M\}$. The model then boils down to a succession of take-it-or-leave-it offers under complete information, which can be solved by backward induction. In the Online Appendix, we show that the following properties obtain in equilibrium: (i) the optimal profit for D_1 accepting an offer (p_0, v_0) is $\pi^* = (\rho - p_0)v_0 - rp_0v_0 + r \min\left(p_0v_0, \sum_{j=1}^{M} \omega_j\right)$; (ii) D_1 accepts the offer if π^* is non-negative, then all offers are made at price ρ; (iii) all dealers but D_1 make zero profit and keep null inventories. D_1 thus extracts the entire surplus of the economy. It is as if D_1 had access to the customers of all further active dealers.

2.2 Rank-free equilibria

We consider perfect Bayesian Nash equilibria. A dealer's expected profit, hence optimal behavior, depends on the probability with which she expects her offer to be accepted. Let us denote $\Phi_{n+1}(p, v)$ the probability that D_{n+1}, if active, accepts an offer (p, v) made by D_n. Using (3) and (4), when accepting (p_{n-1}, v_{n-1}) and making a new feasible offer (p_n, v_n), D_n expects the profit

$$\begin{aligned}\pi_n(p_{n-1}, v_{n-1}, \omega_n, p_n, v_n) = &(\rho - p_{n-1})v_{n-1} - ry_n \\ &+ q\Phi_{n+1}(p_n, v_n)[r \min(p_n v_n, y_n) - (\rho - p_n)v_n].\end{aligned} \tag{5}$$

The profit is the sum of the stand-alone profit and the *expected transaction payoff*, which is equal to D_n's extra payoff if the offer is accepted, times the acceptance probability.

D_n's decision whether to accept an offer, and which new offer to make, could a priori depend on the offers received by dealers prior to D_n and on n itself. However, such information is not payoff-relevant since the asset value is commonly known and the customer demands are independent. We thus focus on *rank-free* equilibria, in which a dealer's strategy only depends on the received offer (p, v) and his type ω but not on n. A dealer's strategy is characterized by an acceptance function A that takes values 1 (acceptance) and 0 (rejection), and a function (P, V) characterizing the dealer's new offer in case of acceptance. An equilibrium is then described by the following conditions:

Definition 1. *The strategy (A, P, V) defines a rank-free equilibrium if, for any $n \geq 1$, any $(p_{n-1}, v_{n-1}, \omega_n) \in \mathbb{R}^{+3}$, the following properties hold:*

(i) The expected acceptance probability is correct: For any offer $(p_n, v_n) \in \mathbb{R}^{+2}$, with $v_n \leq v_{n-1}$, $\Phi_{n+1}(p_n, v_n) = \Pr(A(p_n, v_n, \omega_{n+1}) = 1)$.

(ii) D_n's decisions are optimal, i.e., D_n's offer in case of acceptance $(p_n, v_n) = (P, V)(p_{n-1}, v_{n-1}, \omega_n)$ maximizes D_n's profit (5) given Φ_{n+1} and $A(p_{n-1}, v_{n-1}, \omega_n) = 1$ if and only if this profit is

non-negative.

Note that, although all dealers' equilibrium actions are characterized by the same functions independently of n, as the game unfolds all dealers receive different offers and have different customer demands, so that they behave differently. Moreover, while the equilibrium strategies are rank free, (ii) allows for any deviation, even conditional on the rank.

The sequential nature of the process and the a priori unbounded number of involved dealers allow for a multiplicity of equilibria, depending on the information on which players can condition their strategies. In opaque OTC markets, a dealer typically cannot observe previous offers nor observe at which level of the chain he is, so that strategies are necessarily rank-free. With more information, there are other equilibria, as studied in the Online Appendix. In particular, we show that rank-free equilibria are not only simple but also very natural as a proposer achieves the maximum payoff over all equilibria.

3 Solving the game

We first show some important properties that are satisfied in equilibrium.[12] We then introduce the "collected demand" function, which we use to build all rank-free equilibria.

3.1 Two properties

We first show that a given dealer D_n accepts an offer (p, v) if and only if her customer demand ω_n is larger than a threshold $W(p, v)$. This follows from the fact that D_n can make the same offers independently of her customer demand, and hence dealers with a larger customer demand are necessarily better off. We then derive some properties of $W(p, v)$.

Property 1. *Consider D_n who receives offer (p, v). There exists a unique threshold $W(p, v)$ such that D_n accepts (p, v) if and only if $\omega_n \geq W(p, v)$. We thus have $\Phi_n(p, v) = H(W(p, v))$. If $W(p, v) > 0$ then D_n's profit is null when $\omega_n = W(p, v)$.*

The threshold $W(p, v)$ satisfies the following properties: (i) $W(p, v) = 0$ for $p \leq \frac{\rho}{1+r}$; (ii) $0 < W(p, v) < pv$ for $\frac{\rho}{1+r} < p < \rho$; (iii) $W(p, v) = \rho v$ for $p = \rho$; (iv) $W(p, v) = \infty$ for $p > \rho$.

[12] These properties are satisfied in any equilibrium in which all offers are made at prices below ρ, not only in rank-free equilibria.

Property 1 implies that equilibrium prices are necessarily between $\frac{\rho}{1+r}$ and ρ. These offers are accepted by at least some types of dealers who will have positive (case (ii)) or null (case (iii)) financing needs. Other offers are necessarily suboptimal:

- Case (i): when the price p is sufficiently low then even dealers with a null customer demand accept the offer, D_{n-1} could offer a lower price and keep the same acceptance probability.

- Case (iv): Offers with a price above ρ are never accepted, and are thus dominated by offers below ρ. Intuitively, if D_n makes an offer at a price above ρ and it is accepted, then D_{n+1} has to sell at an even higher price to make a profit. In a rank-free equilibrium, D_n can make the same offer as D_{n+1} and obtain the same acceptance probability, yielding a higher profit than his initial offer. A contradiction.

The next property pertains to the *intermediation revenues* $(\rho - p_n)v_n$. Observe that, along an equilibrium path, if D_n makes an offer to D_{n+1}, it must be the case that D_n makes a positive profit when this offer is accepted. Using (4) and the fact that p_n and p_{n-1} are lower than ρ, we deduce that intermediation revenues decrease along a sequence of dealers:

Property 2. *Along an equilibrium path, for any $n \geq 1$, if D_n receives an offer (p_{n-1}, v_{n-1}) and makes a new offer (p_n, v_n), then D_n's intermediation revenues are lower than D_{n-1}'s: $(\rho - p_n)v_n \leq (\rho - p_{n-1})v_{n-1}$.*

3.2 The collected demand

As the game unfolds, the dealers' customer demands absorb part of the initial volume v_0 bought by D_1, and each dealer sells a lower quantity than the previous one. However, unless there is a strictly positive lower bound on each ω, the maximal number of dealers is not known, because the dealers' demands are variable and unknown. An analysis by induction starting with the last dealer is thus not possible. To find rank-free equilibria, we guess that the optimal offer of a dealer with financing needs y brings an expected payoff equal to $r\Omega(y)$, and we call Ω the *collected demand* function. We show that Ω satisfies a recursive equation with a unique solution, and that it is indeed a function of y only. We then use Ω to solve for the equilibrium offers and acceptance decisions.

The heuristic for the function Ω is the following one. The expected transaction payoff is a function of $v_{n-1} - v_n^C$ and of the financing needs, $y = p_{n-1}v_{n-1} - \rho v_n^C$. Consider an upper bound to this payoff that only depends on y. Let us denote this bound by $rB(y)$. As the transaction payoff is non-decreasing in y for a given offer (from the second line of (5)) the function B can be assumed to be non-decreasing. Now consider dealer D who makes an offer (p, v). The expected transaction payoff is bounded by:

$$qH(W(p,v))[rpv - (\rho - p)v]. \tag{6}$$

Consider a receiver with customer demand $\omega = W(p, v)$. By definition, his profit is non-negative, which, using (5), gives:

$$(\rho - p)v - r(pv - \omega) + rB(pv - \omega) \geq 0. \tag{7}$$

Combining (6) and (7), D's expected transaction payoff is lower than $qH(\omega)r[\omega + B(pv - \omega)]$. Assume that D's offer (p, v) covers at most his financing needs, i.e., $pv \leq y$ (this will be the case in the equilibrium we build below). Since $\omega \leq pv$ (Property 1) and B is non-decreasing, an upper-bound to D's payoff is obtained:

$$rB(y) \leq r \sup_{\omega \leq y} qH(\omega)[\omega + B(y - \omega)]. \tag{8}$$

This leads us to consider the functions B for which the above inequality is binding for any y. Specifically, consider Ω that satisfies the following functional equation:

$$\forall y > 0, \Omega(y) = \sup_{\omega \leq y} qH(\omega)(\omega + \Omega(y - \omega)), \text{ and } \Omega(0) = 0. \tag{9}$$

This equation has the following interpretation: A dealer D with financing needs y chooses a target value ω, and extracts a fixed amount from all types of receivers with customer demand at least equal to ω. This amount corresponds to the transaction payoff divided by r. It is given by the sum of the targeted dealer's customer demand ω plus $\Omega(y - \omega)$, which is the expected value of the customer demand that the targeted dealer, whose needs are equal to

$y-\omega$, will extract from the next dealer. The functional equation (9) ensures that D will indeed collect $\Omega(y)$ if she chooses an optimal value for the target. Due to this interpretation, we call Ω the *collected demand*. It is the counterpart under asymmetric information of $\sum_{j=2}^{N}\omega_j$ in the complete information benchmark.

We prove the following result on Ω:

Theorem 1. *There is a unique function Ω that is bounded, continuous, and satisfies (9). The supremum defined in (9) is reached, and we define the targeted values as:*

$$\mathcal{T}(y) = arg \max_{\omega \leq y} qH(\omega)(\omega + \Omega(y-\omega)). \tag{10}$$

The values in $\mathcal{T}(y)$ have an upper bound independent of y. Moreover, the function Ω is lipshitz with constant q and non-decreasing. The collected demand Ω converges to Ω^∞ as $y \to +\infty$ to the unique value Ω^∞ such that

$$\Omega^\infty = \max_{\omega} qH(\omega)(\omega + \Omega^\infty). \tag{11}$$

Ω cannot be solved in closed-form for a general H. As shown in the Appendix, Ω is obtained as the limit when $n \to +\infty$ of the following sequence of functions:

$$\begin{aligned} \Omega_1(y) &= \max_{\omega \leq y} qH(y)y & (12)\\ \forall n > 1, \Omega_n(y) &= \max_{\omega \leq y} qH(\omega)[\omega + \Omega_{n-1}(y-\omega)]. & (13) \end{aligned}$$

Intuitively, $\Omega_1(y)$ is the customer demand that a dealer with financing needs y can collect from only one other dealer. $\Omega_2(y)$ is the demand that can be collected from a dealer who can also collect from another dealer, etc. The collected demand Ω is the limit of this process, corresponding to the case in which the total number of dealers in a chain is a priori unbounded.

The function Ω is well-behaved as it is lipschitz and increasing. However, it is not necessarily concave and the targeted policy $\mathcal{T}$ may be multi-valued and not increasing, even with non pathological distributions, as we will show in the next section.[13]

[13] This is in contrast with stochastic growth or stochastic consumer problems in which the value function is also defined by a recursive formulation (see, e.g., Stokey and Lucas (1989)). In these problems, the current

Finally, note that the assumption $q < 1$ is essential to guarantee the uniqueness of the function Ω. When $q = 1$, $\Omega(y) = y$ is a solution for any G, but there are possibly others. For example, the sequence Ω_n converges to a function strictly less than y if $g(0) > 0$.

3.3 A rank-free equilibrium

We now use Ω to construct a rank-free equilibrium in which a dealer achieves an expected transaction payoff exactly equal to the upper bound $r\Omega(y)$. As $\mathcal{T}$ may be multi-valued, we define a *selection* of $\mathcal{T}$ as any function T such that $T(y) \in \mathcal{T}(y)$ for any $y \geq 0$.

Theorem 2. *Choose a selection T of $\mathcal{T}$. The following strategies form a rank-free equilibrium. Consider any dealer D receiving an offer (p, v), with customer demand ω and financing needs $y = \max(pv - \omega, 0)$. Denote $\pi^*(p, v, y) = (\rho - p)v - ry + r\Omega(y)$.*

If $\pi^*(p, v, y) < 0$, then D rejects the offer: $A(p, v, \omega) = 0$. This case surely occurs for $p > \rho$.

If $\pi^*(p, v, y) \geq 0$, then D accepts the offer, $A(p, v, \omega) = 1$. For $y = 0$, D makes no further offer and D's profit is $\pi^*(p, v, 0) = (\rho - p)v$. For $y > 0$, D makes a new offer $(P(y), V(y))$ characterized by

$$P(y)V(y) = y \text{ and } (\rho - P(y))V(y) - ry + r(T(y) + \Omega(y - T(y))) = 0. \tag{14}$$

This offer satisfies $P(y) \leq \rho$ and $0 \leq V(y) \leq v - \omega/\rho$. D's expected transaction payoff from the offer is equal to $r\Omega(y)$ and D's profit to $\pi^*(p, v, y)$.

Theorem 2 describes an equilibrium in which D receiving an offer (p, v) with financing needs y expects a transaction payoff equal to $r\Omega(y)$. Hence the profit is $\pi^*(p, v, y)$ and offer (p, v) is accepted only if the profit is non negative, which is the case when the price p is less than ρ and y is low enough. If y is null, we assume to simplify that D makes no further offer.[14] If y is positive, D makes a new offer characterized by two equations (14): (i) the offer

value is related to the discounted *expectation* of the future value. Due to the linearity of the expectation operator, the analysis of the stochastic case is close to that of the deterministic case; in particular, the value function is concave and the policy is increasing. These properties are lost for our function, because the policy corresponds to a threshold that cuts the distribution.

[14]Relaxing this assumption only introduces equilibria in which dealers with null financing needs make

exactly covers D's financing needs; as a result the financing needs of D's target are equal to $y - T(y)$, a quantity that we denote $Z(y)$; (ii) the target's profit is null: The offer she receives is adjusted such that the intermediation revenues, $(\rho - P(y))V(y)$, are equal to her expected cost, $r(Z(y) - \Omega(Z(y)))$, made of the cost without transaction minus the presumed expected transaction payoff. As a result, it is weakly optimal for her to accept $(P(y), V(y))$ so that $T(y)$ is indeed the threshold associated to the offer $(P(y), V(y))$. The recursive equation (9) satisfied by Ω then ensures that D's behavior is optimal and that D achieves a transaction payoff equal to $r\Omega(y)$. Finally, the inequalities $0 \leq V(y) \leq v - \omega/\rho$ imply that if the offer is accepted, D has enough units to serve the amount offered to the next dealer and his customer demand. Hence, although our game only requires $V(y) \leq v$, the customers are surely fully served in this equilibrium when possible.

The trade-offs on the choice of the target are easy to describe when the objective of the program (9) is differentiable at the target. Assume that the distribution G admits a density and Ω is differentiable at $y - T(y)$. The target $\omega = T(y)$ must satisfy the first order conditions:

$$-\frac{g(\omega)}{H(\omega)} + \frac{1 - \Omega'(y - \omega)}{\omega + \Omega(y - \omega)} \geq 0, \text{ with an equality if } \omega < y. \tag{15}$$

The marginal impact of ω is decomposed into two effects: a decrease in the probability of acceptance and an increase in D's transaction benefit in case of acceptance, as the target has more customer demand, i.e., $1 - \Omega'(y - \omega) \geq 0$.

Properties of offers in rank-free equilibria. From (14), the equilibrium offer $(P(y), V(y))$ of a dealer with financing needs y is given by:

$$P(y) = \frac{\rho}{1 + r\frac{Z(y) - \Omega(Z(y))}{y}} \text{ and } V(y) = \frac{y + r(Z(y) - \Omega(Z(y)))}{\rho}, \tag{16}$$

where $Z(y) = y - T(y)$ are the financing needs of the target. We analyze how these offers, the offering dealer's profit, and the receiving dealer's inventory depend on the financing needs y and the funding spread r. These properties will be useful to discuss the empirical implications of the model in Section 5.

unnecessary offers that are only accepted by other dealers who have zero financing needs, etc., forming artificially long sequences that do not change any player's payoff.

Proposition 1. *1. Consider a dealer offering $(P(y), V(y))$. The volume is increasing in y and ρ and decreasing in r. The price is increasing in ρ and decreasing in r, but not necessarily monotonous in y. In the limit for high financing needs:*

$$\lim_{y \to +\infty} P(y) = \frac{\rho}{1+r}, \; V(y) \underset{+\infty}{\sim} \frac{(1+r)y}{\rho}. \tag{17}$$

The profit $\pi^*(p, v, y)$ is decreasing in y and r, and increasing in ρ.

2. Consider the receiver of offer $(P(y), V(y))$ with customer demand $\omega \geq T(y)$. Her profit is decreasing in her financing needs $z = \max(y - \omega, 0)$ and given by

$$\pi^*(P(y), V(y), z) = r[Z(y) - \Omega(Z(y)) - (z - \Omega(z))]. \tag{18}$$

Her inventory is decreasing in z and given by

$$v^I = V(y) - \frac{\omega}{\rho} - V(z) \text{ if her offer is accepted and } v^I = V(y) - \frac{\omega}{\rho} \text{ otherwise.} \tag{19}$$

The results on the proposer follow straightforwardly from the expressions (14) and the fact that $z - \Omega(z)$ is non-decreasing in z. The intuition for the effect of r is that a higher r makes the receiver lose more on each unit she buys if her own offer is not accepted. Hence, she requires a lower price to accept an offer. Conversely, the proposer loses more if his offer is not accepted and is hence ready to concede larger intermediation revenues. Hence, the proposer optimally chooses to sell more units at a lower price. The limit values follow from the fact that Z and Ω are bounded so that $\frac{Z(y) - \Omega(Z(y))}{y}$ tends to 1.

Consider now the receiver's profit (18). From the second equation in (14), we know that the intermediation revenues, $(\rho - P(y))V(y)$, are exactly equal to the target's expected cost, $r(Z(y) - \Omega(Z(y)))$. Then, each unit of customer demand brings an additional profit, as long as $\omega < y$ (i.e., $z > 0$). Due to asymmetric information, customer demand provides an informational rent for those dealers who receive an offer and have more than the targeted level. This rent is equal to the difference in expected costs with the target, which are proportional to r. As for the receiver's inventory, the receiver buys $V(y)$ units, sells $\min(V(y), \frac{\omega}{\rho})$ units to her customers and $V(z)$ units to the next dealer if z is positive. This gives (19).

A surprising finding in Proposition 1 is that the price is not necessarily monotonous in y. To understand why, it is useful to relate the intermediation revenues $(\rho - P(y))V(y)$ with the financing needs $Z(y)$ of the target. Since the intermediation revenues are exactly equal to the target's expected cost $r(Z(y) - \Omega(Z(y)))$, they vary with y as $Z(y)$. Since $V(y)$ is increasing in y, if in addition Z is decreasing then the price necessarily increases. However, Z is likely to be increasing. In particular, this is true under a standard assumption on G:[15]

Lemma 1. *Assume that G satisfies the following condition:*

$$G \text{ admits a continuous density } g, \text{ with an increasing hazard rate } g/(1-G). \tag{A1}$$

Then there exists $\underline{y}_1 > 0$ such that the financing needs of a targeted dealer, $Z(y) = y - T(y)$, are null for $y < \underline{y}_1$. For $y > \underline{y}_1$, Z is increasing in y and the target has a positive lower bound.

From the discussion above, we immediately deduce the following result.

Corollary 1. *Under (A1), the intermediation revenues $(\rho - P(y))V(y)$ increase in y.*

Thus, under (A1), we obtain the intuitive result that a dealer with higher financing needs y chooses a target with higher financing needs $Z(y)$. As a result, the target has higher financing costs, and must be compensated for these costs by higher intermediation revenues. To offer higher intermediation revenues, a dealer can offer a lower price, a higher volume, or both. With a divisible asset, the direction in which each variable is adjusted is not straightforward. To better understand how the price and volume are affected by the financing needs, it is useful to consider two examples.

Example: Degenerate distribution. Consider the case in which all dealers face a sure demand $\overline{\omega}$ from their customers[16] (the distribution G is degenerate). In that case, the target

[15] Standard distributions (uniform, beta, or exponential) satisfy (A1). The assumption is standard in auction theory.

[16] This special case is related to Viswanathan and Wang (2004), with an important difference: In their paper, the total number of dealers is known, and the game can be solved by backward induction. In our model, when there is no uncertainty the total number of dealers is endogenous and is a function of D_1's financing needs. When there is uncertainty, the total number of dealers becomes stochastic.

is always $\overline{\omega}$ and the function Ω satisfies the following equation:

$$\Omega(y) = \begin{cases} q[\overline{\omega} + \Omega(y - \overline{\omega})] & \text{for } y > \overline{\omega} \\ qy & \text{for } y \leq \overline{\omega}. \end{cases} \tag{20}$$

We can explicitly solve for Ω, which is piece-wise linear and concave:

$$\Omega(y) = q[1 + q + q^2 + ... + q^{n-1}]\overline{\omega} + q^{n+1}(y - n\overline{\omega}) \text{ for } n\overline{\omega} < y \leq (n+1)\overline{\omega}. \tag{21}$$

Using (16), we see that $P(y)$ varies with y like $\frac{y}{y - \overline{\omega} - \Omega(y - \overline{\omega})}$, which is decreasing in y.[17] There is no asymmetric information in this example. However, the price is not constant because the target faces higher expected financing costs when y is higher.

Example: Binary and Gamma distributions. When the distribution is not degenerate, the price can be increasing at points where $Z(y)$ is increasing (Proposition 1). To illustrate this point, Fig. 2 shows plots of T, Ω, and $P(y)$ with two different distributions, which we use for illustration throughout the paper.[18]

Both distributions have the same mean but give rise to different targets T. In particular, in the binary case there is a value of $\bar{y}$ such that both values of $\tilde{\omega}$ are in the target $\mathcal{T}(y)$. As y crosses the threshold $\bar{y}$, the target makes a jump upwards and the price increases discontinuously. With the Gamma distribution instead prices are monotonously decreasing in y, despite the target being non-monotonous (Z is monotonously increasing in this example). In both cases, in the limit the price converges to a minimum when y becomes very large, and is equal to ρ when financing needs are sufficiently small. Hence, the overall shape of P is decreasing, but it can increase on a finite number of intervals (or at a finite number of discontinuity points).

[17] The derivative of the ratio has the sign of $\Omega'(y - \overline{\omega})y - \overline{\omega} + \Omega(y - \overline{\omega})$, which is negative: Because Ω is concave, $\Omega(0) - \Omega(y - \overline{\omega}) \leq -\Omega'(y - \overline{\omega})(y - \overline{\omega})$. Using $\Omega(0) = 0$ and $\Omega'(y) \leq 1$ gives the result.

[18] Unless explicitly mentioned, the parameters used in all figures are equal to $\rho = 100$, $r = 0.05$, $q = 0.9$. The distribution G is either "Gamma", a Gamma distribution with parameters $k = 4.65$ and $\theta = 1,000$, or "Binary", a binary distribution with $\Pr(\omega = 2,385) = 0.05$ and $\Pr(\omega = 4,770) = 0.95$.

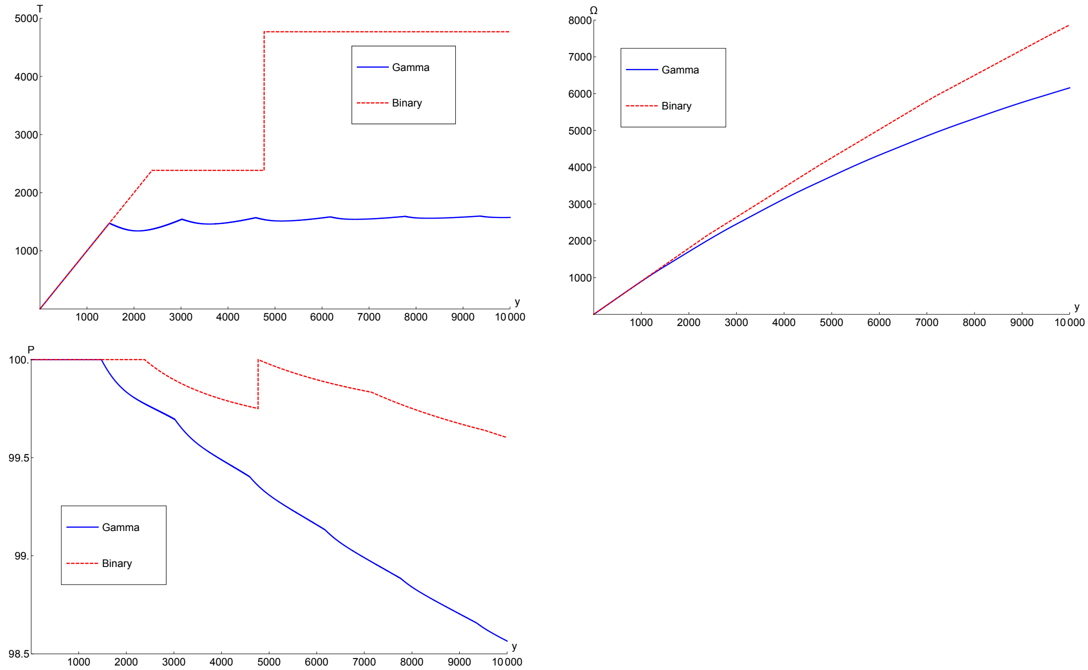

Figure 2 – Equilibrium target $T(y)$ (top left), collected demand $\Omega(y)$ (top right), and offered price $P(y)$ (bottom left), for two distributions G.

4 Asset Dissemination

This section derives some results on asset dissemination in our model. We distinguish two notions of dissemination: the number of dealers involved in selling the asset ("dissemination length") and the distribution of inventories across dealers and customers.

Let us first illustrate these notions with an example, using the same parameters as in Fig. 2. We start with an offer (p_0, v_0), with $p_0 = \frac{\rho}{1+r} = 95.24$ so that D_1 surely accepts it and $v_0 = 105$. The dissemination process then depends on which dealers are active and on the demand of their customers. Consider for example three active dealers and $(\omega_1, \omega_2, \omega_3) = (2384, 4769, 4769)$. The sequence of offers and their acceptance or rejection are determined by repeatedly applying Theorem 2 to the sequence of dealers' financing needs. Fig. 3 shows the outcome: three dealers are involved in disseminating the asset, and the initial volume 105 is endogenously distributed across the first two dealers and the customers of all three.

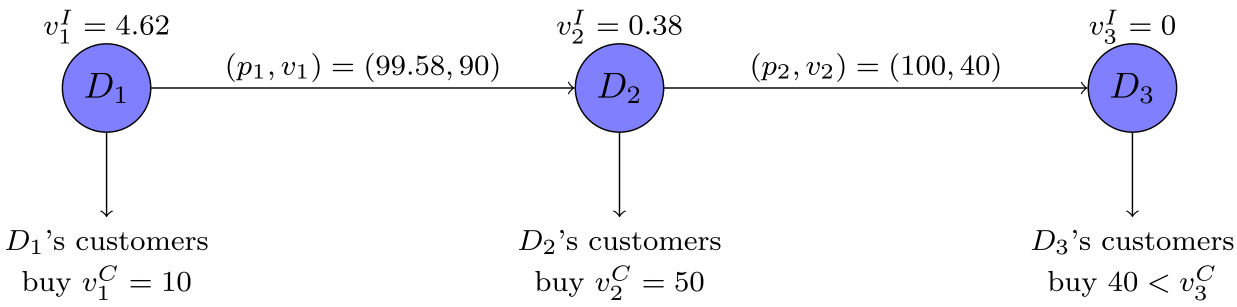

Figure 3 – Dissemination with three dealers. The graph shows the offer (p_n, v_n) made by each active dealer D_n, the inventory v_n^I he keeps, and the volume sold to his customers.

4.1 Dissemination length

Let us formally define dissemination length. Consider dealer D_1 who accepts an offer and has financing needs $y > 0$, so that he also makes a new offer (customer demand being independent across dealers, any dealer with the same financing needs y can be relabeled as D_1). If D_1's offer is accepted, we denote M the last active dealer and σ the sequence $(\omega_2, ..., \omega_M)$, which we call "dealer shocks". There exists $N \leq M$ such that the last dealer to accept an offer is the N-th (either he does not make an offer himself, or his offer is rejected). We define dissemination length as N and denote it $N_R(y, \sigma)$. In the special case in which D_2 is inactive we have $\sigma = \emptyset$ and we define $N_R(y, \sigma) = 1$.

Dissemination length is a random variable. We analyze two statistics: (i) *expected length*, $\mathbb{E}(N_R(y, .))$, is the expectation of dissemination length; (ii) *maximum length*, equal to $\max_\sigma N_R(y, \sigma)$, is the maximum value of dissemination length across all dealer shocks.

Maximum length can be obtained by looking at dealer shocks such that each dealer after D_1 is active and has the exact customer demand targeted by the previous dealer. We call such a sequence of dealers the *targeted sequence*. By construction, each dealer in the sequence makes an offer, except if her financing needs are null. Formally, the financing needs of D_2 (D_1's target) are $Z(y) = y - T(y)$ and D_2's target is $T(Z(y))$. Recursively, as long as the financing needs of the $(k-1)$ first dealers are positive, the financing needs of the k-th dealer are $Z^{k-1}(y)$. If $Z^{k-1}(y)$ is positive, D_k makes an offer, which targets $T(Z^{k-1}(y))$. Under (A1), targets have a positive lower bound (Lemma 1). As $Z^k(y)$ is a decreasing sequence, there exists a smallest integer n such that $Z^{n-1}(y) = 0$. By definition, the n-th dealer has zero financing needs and hence makes no offer. Thus, the dissemination length of the targeted sequence is equal to n. We call n the *targeted length* and denote it $N_T(y)$. The targeted length increases with y since Z increases.

Realized and targeted sequences differ for two reasons. First, some offers are not accepted, either because the receiver is inactive or because she lacks customer demand. Second, a receiver accepting an offer has a larger customer demand than the target, and hence lower financing needs. An implication is that the targeted length is equal to the maximum length.

Proposition 2. *Under (A1), the maximum length is equal to the targeted length $N_T(y)$. It is finite and non-decreasing in y.*

Fig. 4 plots the targeted length $N_T(y)$, both for a Gamma distribution that satisfies (A1), and for a discrete distribution that does not. In the former case, the length is monotonous in y, as expected, whereas in the latter case it is not. As we see on Fig. 2, this is because $T(y)$ can increase very sharply. In particular, when y reaches 4770 it is optimal to have $T(y) = y$, that is, to make an offer at price ρ, which gives a targeted length of 1.

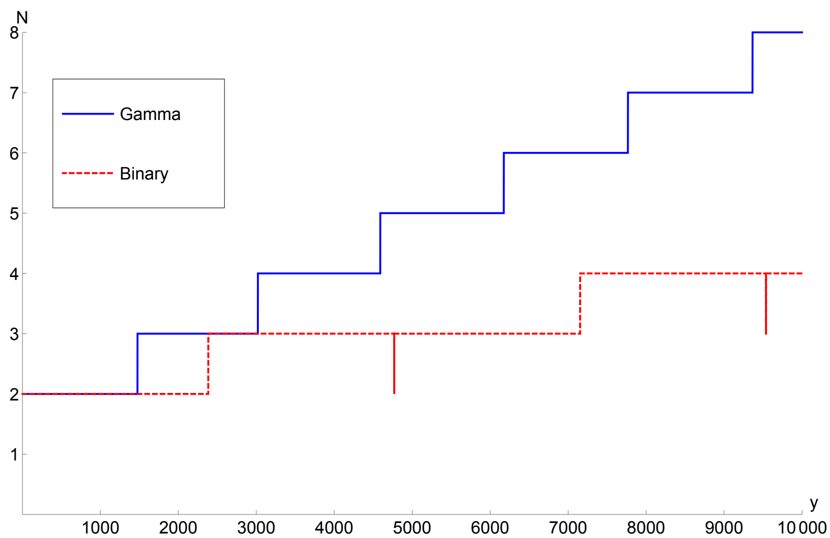

Figure 4 – Targeted length $N_T(y)$, for two distributions G.

We can derive further analytical results on the expected length and the maximum length by considering the asymptotic case $y \to +\infty$.[19]

Lemma 2. *Under (A1) the target $T(y)$ converges to ω^∞ as $y \to +\infty$.*

[19]The proof uses a weaker assumption than (A1) that also admits discrete distributions.

Since the target has an upper bound, when the financing needs of a dealer become infinitely large, the financing needs of his target are also infinitely large, as well as the financing needs of the following target, etc. Hence, (9) reduces to (11) and the limit value ω^∞ of the target solves a simple fixed point problem. Moreover, at each step the game continues with probability $qH(\omega^\infty)$, and stops otherwise. Hence, the expected length converges to ℓ_{ω^∞}, with:

$$\ell_\omega = \frac{1}{1-qH(\omega)}. \tag{22}$$

4.2 Distribution of inventories

We now consider our second notion of dissemination and define how many units of the asset dealers keep on inventory and how much their customers buy at each step of the process. Consider an offer (p_0, v_0) accepted by D_1 with demand ω_1 and the dealer shocks $\sigma = (\omega_2..., \omega_N)$. All dealers except possibly D_N serve their customers in full. Since the customers of D_n demand $v_n^C = \frac{\omega_n}{\rho}$, dealers' inventories write as:

$$\begin{aligned}
v_1^I &= v_0 - v_1^C - V(y_1), \\
v_n^I &= V(y_{n-1}) - v_n^C - V(y_n), n \in [2, N-1] \\
v_N^I &= \max(V(y_{N-1}) - v_N^C, 0)
\end{aligned}$$

where V is given by (16). Summing these equations we have:

$$\underbrace{\sum_{n=1}^{N-1} v_n^C + \min(v_N^C, V(y_{N-1}))}_{\text{Customer purchases}} + \underbrace{\sum_{n=1}^{N} v_n^I}_{\text{Dealer inventories}} = v_0. \tag{23}$$

For a given dealer D_1 who accepts offer (p_0, v_0) and has financing needs y, we define the *customer purchases* as $C_R(p_0 v_0, \omega_1, \sigma) = \sum_{n=1}^{N-1} v_n^C + \min(v_N^C, V(y_{N-1}))$. Equation (23) means that dealer inventories are simply the initial volume minus the customer purchases.

4.3 Effect of asymmetric information

We now study the effect of asymmetric information by comparing dissemination length in the baseline game and in the complete information benchmark. Consider D_1 who has accepted offer (p_0, v_0), has financing needs y, and take σ as given. As we show in the Online Appendix A, in the complete information benchmark the dealers who buy the asset are all the active dealers necessary to finance $p_0 v_0$ if $p_0 v_0 < \sum_{j=1}^{M} \omega_j$, or all the M active dealers otherwise. This defines the dissemination length in the complete information benchmark, denoted $N_F(y, \sigma)$. In the baseline game, if the dealer of rank $N_F(y, \sigma)$ is reached then either he is the last active dealer or he has null financing needs. In either case the game stops. Hence, we have $N_R(y, \sigma) \leq N_F(y, \sigma)$, with a strict inequality when σ is such that an active dealer rejects an offer that would have been accepted under complete information. Similarly, we define $C_F(p_0 v_0, \omega_1, \sigma)$ the customer purchases under complete information and obtain that $C_R(p_0 v_0, \omega_1, \sigma) \leq C_F(p_0 v_0, \omega_1, \sigma)$. Thus, starting with a given dealer, asymmetric information unambiguously reduces the extent of dissemination.

Proposition 3. *For a given D_1 accepting offer (p_0, v_0) and for any ω_1 and σ, asymmetric information reduces both the dissemination length and the customer purchases: $N_R(y, \sigma) \leq N_F(y, \sigma)$ and $C_R(p_0 v_0, \omega_1, \sigma) \leq C_F(p_0 v_0, \omega_1, \sigma)$.*

In order to quantify the difference between $N_F(y, \sigma)$ and $N_R(y, \sigma)$, one can compute their expected values recursively. In the baseline game, starting with D_1 with financing needs y, there is a probability $qH(y)$ that the next dealer accepts D's offer and has zero financing needs, hence finishing the process. With probability $q[H(T(y)) - H(y)]$, the next dealer accepts D's offer and we can compute the expected length from then onwards, conditional on the financing needs $y - \omega$ of this dealer. We obtain the following recursive expression:

$$\mathbb{E}(N_R(y, .)) = 1 + qH(y) + q \int_{T(y)}^{y} \mathbb{E}(N_R(y - \omega, .)) dG(\omega). \tag{24}$$

Similarly, $N_F(y,\sigma)$ satisfies the same formula, replacing $T(y)$ with 0:[20]

$$\mathbb{E}(N_F(y,.)) = 1 + qH(y) + q\int_0^y \mathbb{E}(N_F(y-\omega,.))dG(\omega). \tag{25}$$

The difference between $\mathbb{E}(N_F(y,.))$ and $\mathbb{E}(N_R(y,.))$ is thus driven by the level of the target $T(y)$, and its iterations along the possible sequences of dealers.

Finally, we can offer some closed-form quantification of the impact of asymmetric information by studying a degenerate distribution benchmark. For a given distribution G with mean $\bar{\omega}$, we consider the degenerate distribution $\widehat{G}$ in which all dealers have the certain demand $\bar{\omega}$. We denote $\mathbb{E}(\widehat{N}_R(y,.))$ and $\widehat{N}_T(y)$ the expected length and targeted length under distribution $\widehat{G}$, respectively.

Proposition 4. *Under (A1) we have the following limits:*

$$\lim_{y\to+\infty} \mathbb{E}(N_R(y,.)) = \ell_{\omega^\infty} \quad \lim_{y\to+\infty} \mathbb{E}(\widehat{N}_R(y,.)) = \ell_0 = \tfrac{1}{1-q} \tag{26}$$

$$N_T(y) \underset{+\infty}{\sim} \frac{y}{\omega^\infty} \qquad\qquad \widehat{N}_T(y) \underset{+\infty}{\sim} \tfrac{y}{\bar{\omega}} \tag{27}$$

(26) shows that uncertainty about customer demand leads to lower dissemination length in expectation, at least for large enough financing needs. The reason is the same as in Proposition 3: uncertainty introduces an incentive to make offers that are rejected, in which case dissemination is interrupted. (27) instead implies that the impact of uncertainty on the targeted length and hence on maximum length is ambiguous, and depends on whether the asymptotic targeted type ω^∞ is larger than the average type $\bar{\omega}$. Our simulations show that both are possible: with the Gamma distribution we have $\omega^\infty \leq \bar{\omega}$, whereas with the discrete distribution we have the opposite.

5 Implications

In this section we derive several implications on dissemination, inventories, and the terms of interdealer transactions. We then discuss how they relate to the empirical literature.

[20](25) is close to what is known in renewal theory as a "renewal equation". In contrast, (24) is more complicated because one of the bounds of the integral is endogenous.

5.1 Comparative statics

The funding spread. We derive some comparative statics on dissemination and dealers' profits with respect to the funding spread r. We start with D_1 with customer demand ω_1 who accepts an offer (p_0, v_0) at a given spread r. Consider a decrease in r. Since D_1's profit is decreasing in the spread (Proposition 1), D_1 still accepts the offer when the spread is lowered. Theorem 2 shows that T does not depend on r. Hence, the set of dealers who accept D_1's offer is unchanged and consists in all types $\omega_2 \geq T(y)$. We can then repeat the argument: D_2's target is not affected by r, etc. We obtain that for any dealer shocks $\sigma = (\omega_2, \omega_2 ..., \omega_M)$ up to the last active dealer M, the realized sequence is unaffected by r. The same holds for the targeted sequence starting with D_1, as it only depends on $Z(y) = y - T(y)$, which does not depend on r. We deduce the following implication:

Implication 1. *Consider D_1 who accepts an offer (p_0, v_0) for a given r. A decrease in r leaves unchanged the realized and targeted sequences starting with D_1, the acceptance probability of the offers made, and the financing needs of all dealers following D_1.*

Thus a decrease in r does not affect dissemination length as long as D_1 accepts the offer (p_0, v_0). It has however an impact on the distribution of inventories. Consider first the total dealer inventories and customer purchases. A decrease in r impacts them through the volume $V(y_{N-1})$ bought by the last dealer, as can be seen from (23). Since the financing needs are unaffected by r, this volume decreases when r decreases (equation (16)). Hence, either the total dealer inventories are higher and customer purchases lower, or both are unchanged. Second, consider the distribution of inventories among dealers. A decrease in r leads D_1 to offer a lower volume, hence to increase his inventory. Any other dealer D_n receives fewer units from D_{n-1} and sells fewer units to D_{n+1}. Under (A1), it follows from Lemma 1 that the combined effect is a decrease of D_n's inventory. These findings are summarized in the following implication.

Implication 2. *Consider D_1 who accepts an offer (p_0, v_0) for a given r. For any dealer shocks σ, a decrease in r leads to higher prices and lower volumes offered by each D_n, higher or unchanged total dealer inventories, and lower or unchanged customers purchases. D_1's inventory increases and, under (A1), any subsequent dealer D_n's inventory decreases.*

We conclude by considering how the funding spread affects dealers' profits. From Proposition 1, we know that the funding spread provides a rent to intermediaries. When r decreases, their rent is decreased, leaving larger profits to the first dealer. Specifically, (18) yields the following implication.

Implication 3. *Consider D_1 who accepts an offer (p_0, v_0) for a given r. For any dealer shocks σ, a decrease in the funding spread r leads to an increase in D_1's profit and a decrease in the subsequent dealers' profits.*

The asset's value. Consider assets with different values of ρ but identical distributions G for ω. Since $\omega = \rho v^C$, this means that the amount of cash that the customers are ready to invest in these assets are identical. Then T, Z, and Ω are unaffected by ρ since they only depend on G (Theorem 2). This implies that D_1's profit is increasing in ρ for a given (p_0, v_0) (Proposition 1). The same reasoning as in Implication 1 shows the following result.

Implication 4. *Consider D_1 who accepts an offer (p_0, v_0) for a given ρ. Consider assets with larger values for ρ and identical distributions G. The realized and targeted sequences starting with D_1, the acceptance probability of the offers made, and the financing needs of all dealers following D_1 are identical.*

Using this result, the fact that Z and B only depend on G, and expression (16), we straightforwardly obtain the following implication.

Implication 5. *Consider assets with different values of ρ and identical distributions G. The rebate offered by a dealer with financing needs y is equal to:*

$$\frac{\rho - P(y)}{\rho} = 1 - \frac{1}{1 + r\frac{Z(y) - \Omega(Z(y))}{y}}. \tag{28}$$

This measure is independent of ρ and only depends on the financing needs of the proposer and the expected costs of the target.

The quantity $\frac{\rho - P(y)}{\rho}$ measures the relative price concession made by a dealer D with financing needs y. As it is independent of ρ, it can be compared across assets. Recall that D's offer necessarily satisfies $P(y)V(y) = y$. Expression (28) clarifies the determinants of

D's bargaining power. When the expected costs of the target $r[Z(y) - \Omega(Z(y))]$ are low relative to D's financing needs, D is able to make an offer at a high price and a low volume. Conversely, when the target's costs are high relative to y then D has to make an offer with a low price and a high volume for the target to accept.

Other parameters. Unlike ρ and r, the other parameters of the model, q and G, do not leave the functions T and Ω unchanged. As a result, without more assumptions the comparative statics on these parameters are ambiguous. For instance, an increase in the probability q that an intermediary is active makes it more likely that a dealer will find a counterparty. For a given T this increases dissemination length. However, a higher q also decreases the target's expected costs, which can incentivize a dealer to choose a higher target T, which decreases dissemination length.

5.2 Correlations between dissemination and terms of trade

In the model asset dissemination and the prices and quantities offered are jointly determined. This section discusses the empirical correlations that this joint determination implies. To see the mapping between our model and the data observed by empiricists, it is useful to consider an example, using the same parameters as in Fig. 2 and the Gamma distribution. Starting with offer $(95.24, 105.0)$, we simulate the model 1,000,000 times and record the average price and volume offered at each rank in a realized sequence, as well as the number of sequences of each length (the last line). Table 1 reports the results for the sequences with length between 1 and 5 (there are also 100,020 sequences of length 0 and 14 sequences of length 6).

As Table 1 illustrates, empiricists who would observe the data generated by our model could analyse (i) how offers evolve *along a sequence*, by comparing different lines for a given column; and (ii) how dealers' offers differ *across sequences* of different lengths, by comparing different columns for a given row.

Analysis along a sequence. We first consider how prices and volumes evolve along a realized sequence of dealers D_1, D_2, etc. We know from Property 2 that, along the sequence, the financing needs y_n, the volume v_n, and the intermediation revenues $(\rho - p_n)v_n$ decrease. The prices p_n may not be monotonic, although they tend to increase with n and become closer to the fundamental value. This non-monotonicity occurs because dealers with larger

Offer / Size	1	2	3	4	5
(p_0, v_0)	(95.24,105.0)	(95.24,105.0)	(95.24,105.0)	(95.24,105.0)	(95.24,105.0)
(p_1, v_1)	End	(99.42,44.9)	(99.14,71.7)	(98.96,72.9)	(98.84,81.0)
(p_2, v_2)		End	(99.81,21.0)	(99.47,41.8)	(99.23,55.7)
(p_3, v_3)			End	(99.94,11.5)	(99.66,30.7)
(p_4, v_4)				End	(99.98,6.90)
(p_5, v_5)					End
# Sequences	131,362	340,695	356,243	69,095	2,571

Table 1 – Simulation results - Average offers along sequences of different lengths.

The table reads as follows: the cell $(99.14, 71.7)$, second row and third column, is based on the offers made by D_1 in the 356,243 realized sequences of length 3. Averaging over these sequences, the second dealer made an offer with an average volume of 71.7, for an average price of 99.14.

financing needs may adjust their offer either by increasing the volume or decreasing the price (see the discussion after Implication 5) and would not occur with an indivisible asset.

Analysis across sequences. Our model clarifies that chains of different lengths are associated with different prices and volumes. Starting with the same initial offer, different chains obtain due to variations in the dealer shocks. As reported in Table 1, longer chains tend to be associated with lower prices: dissemination length is longer when dealers have less customer demand, which is also associated with higher financing needs, higher intermediation revenues, and lower prices.

Effect of the initial offer. The previous discussion considers variations in chains due to dealer shocks, keeping the same initial offer (p_0, v_0). Conversely, variations in the initial offer will have an impact both on dissemination length and prices. Analytically, we can deduce from Corollary 1 and Proposition 2 the following result.

Implication 6. *Under (A1), for a given price p_0, a lower initial volume v_0 leads both to lower intermediation revenues and lower targeted lengths.*

For a given price p_0, a lower initial quantity v_0 to disseminate leads to lower financing needs everywhere along any chain. It then takes fewer dealers to disseminate the asset. Since their financing needs are lower they optimally choose tougher offers with lower intermediation revenues. If this is done by increasing the price then we obtain a negative relation between the dissemination length and the price of the asset, as illustrated on Fig. 5.

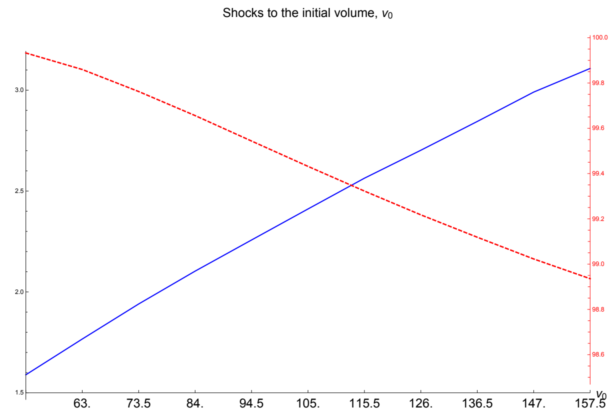

Figure 5 – Relation between expected length and prices. The figure plots the expected length (blue solid line, left axis) and the average price weighted by volume (red dotted line, right axis) over 100,000 simulations, for different parameterizations. The baseline parameters are as in Fig. 2 with the Gamma distribution and we vary the initial volume v_0.

5.3 Links with the empirical literature

We now briefly review how our implications relate to facts documented in the empirical literature, or are yet to be tested.

Proposition 1 predicts a negative relationship between the funding conditions of dealers (parameterized by r) and prices. This has been documented for instance in Garleanu and Pedersen (2011) and in Bao, O'Hara, and Zhou (2018). In addition, Implications 2 and 3 suggest to test the impact of r on interdealer trading volumes, dealers' trading profits, and inventories, controlling for whether a given dealer starts a sequence of interdealer trades.[21]

Our results on correlations between terms of trade and dissemination are connected to several recent empirical papers. A difficulty is that, in order to identify transactions belonging to the same sequence, these papers make different assumptions and thus identify different types of "intermediation chains", that have similarities with the dissemination process in our model but also some important differences.

On the market for securitizations, Hollifield, Neklyudov, and Spatt (2017) identify chains in which each dealer buys assets from an initial customer or from another dealer, and sells to his customers and at most one other dealer. These chains closely resemble those obtained in our model. However, in order to make sure that the transactions they observe belong to the

[21] A dealer profit on a given trade may be difficult to compute without information on inventory costs. Inventories can be inferred from transaction data, as in, e.g., Choi, Shachar, and Shin (2019).

same chain, they focus on series of transactions that leave each dealer with a null inventory. On the U.S. corporate bond market, Friewald and Nagler (2019) identify more general chains in which a dealer can sell to several customers and several other dealers at the same time. They report that two thirds of the chains are incomplete, meaning that dealers do keep some inventory. On the municipal bond market, Li and Schürhoff (2019) apply a more restrictive criterion and focus on "no-split chains" in which a dealer sells the exact quantity he bought to either a dealer or to another customer, but does not split between the two.

While Section 6 below considers variants of the model that can fit these different types of chains, one has to keep these differences in mind when applying our results. Regarding results "across sequences", combining Proposition 3 with the fact that the price is equal to the asset value ρ under complete information means that asymmetric information generates both lower prices and lower dissemination length in our model, and can thus explain a positive correlation between prices and length. Conversely, Implication 6 is more in line with a negative correlation. Friewald and Nagler (2019) give evidence for a positive correlation on the U.S. corporate bond market (aggregating prices in dealer-to-dealer and customer-to-dealer transactions). Table 9 in Hollifield, Neklyudov, and Spatt (2017) focuses on interdealer transactions and reports that interdealer spreads decrease in chain length, which goes in the same direction. Table XII in Li and Schürhoff (2019) contains an analysis "along a sequence" and shows that interdealer prices do not move monotonically with the rank of a dealer. Property 2 suggests that when the volume can be split, as in Hollifield, Neklyudov, and Spatt (2017), an analysis along a sequence should use intermediation revenues. The property predicts that those should decrease monotonically along the sequence.

6 Extensions

In this section we show how our approach can be extended to study other forms of intermediation on OTC markets.[22] We first show how to account for the different types of intermediation chains that have been documented in the literature, and then consider a variant of the model that can be applied to the repo market to study rehypothecation.

[22] For brevity we do not include the proofs for these different extensions. They are all simple variants on the proof of Theorem 2.

6.1 The structure of intermediation chains

We consider three variants of our model that can fit some special types of chains documented in the literature: (i) dealers keeping a null inventory (as in Hollifield, Neklyudov, and Spatt (2017)); (ii) dealers selling the entire volume to other dealers (as in Li and Schürhoff (2019)); (iii) dealers selling to several other dealers (as in Friewald and Nagler (2019)). We also show how to endogenize the first customer-to-dealer transaction.

Inventory costs. In our model dealers have no inventory costs. As a result, they sell just enough of the asset to cover their financing needs, and prefer keeping the rest in inventory to selling it at a discount. We now show that if instead dealers face inventory costs but no financing costs they optimally keep a null inventory.

Assume r is null, but a dealer D_n ending with an inventory v_n^I has to pay a cost cv_n^I, with $c > 0$. The setup is otherwise identical to our initial model. Let D_n accept offer (p_{n-1}, v_{n-1}), and denote $y_n^I = \max(v_{n-1} - v_n^C, 0)$ the "offloading needs" of D_n, that is, his inventory if he doesn't trade with D_{n+1}. If he makes an offer (p_n, v_n) with $v_n \leq y_n^I$, D_n's payoff is:

$$(\rho - p_{n-1})v_{n-1} - cy_n^I \quad \text{if } D_{n+1} \text{ rejects } (p_n, v_n) \tag{29}$$

$$(\rho - p_{n-1})v_{n-1} - (\rho - p_n)v_n - c(v_n^I - v_n) \quad \text{if } D_{n+1} \text{ accepts } (p_n, v_n). \tag{30}$$

We then obtain the same model as in the previous sections by replacing the value of the demand ω_n by the number of units v_n^C, the financing needs y_n by the offloading needs y_n^I and the financing costs ry_n by the inventory costs cy_n^I. Transposing Theorem 2, a rank free equilibrium is characterized by the collected demand Ω^I defined by the following equation[23]:

$$\forall y^I > 0 : \Omega^I(y^I) = \sup_{v^C \leq y^I} qH(v^C)(v^C + \Omega^I(y^I - v^C)), \text{ and } \Omega^I(0) = 0 \tag{31}$$

and the associated targets T^I. When a dealer makes an offer targeting $T(y^I)$, the offer $(P^I(v^I), V^I(v^I))$ is characterized by

$$V^I(y^I) = y^I \text{ and } (\rho - P^I(v^I))V^I(y^I) - cy^I + c(T^I(v^I) + \Omega^I(y^I - T(y^I))) = 0. \tag{32}$$

[23] H^I corresponds here to the distribution of v^C instead of ρv^C.

In particular, the first condition says that it is optimal for the dealer to sell all his units of the asset. This corresponds in the previous model to the property that the value of an optimal offer is equal to the proposer's financing needs. This implies that, *on the equilibrium path*, all dealers keep a null inventory, except possibly at the last step if the last offer is rejected.

This variant of the model explains the existence of chains in which dealers keep no inventory. The reason for this property is that, conditionally on a transaction occurring between D_n and D_{n+1}, keeping an extra unit of inventory costs c to D_n, while taking an extra unit costs only $c[1 - \Omega'(y^I_{n+1})]$ to D_{n+1}, due to the option for D_{n+1} to pass on the inventory to another dealer. Hence, for a given target $T^I(y^I_n)$, it is always optimal for D_n to pass on as much inventory as possible, and compensate his target by offering a lower price.

Pure intermediaries and no-split chains. In our model, a dealer who buys a given quantity never sells the exact same quantity to another dealer. To generate no-split chains, we consider the following extension: in addition to regular dealers, there are also (inter-dealer) *intermediaries*, who do not have customers themselves but can contact one additional dealer. We assume that a dealer D_n first makes an offer to a regular dealer. If this offer is rejected, with probability q_b the dealer finds an intermediary and makes a new offer. The intermediary is then in the same situation and can make an offer to another dealer, or in case of rejection to another intermediary. Importantly, there is no asymmetric information when making an offer to an intermediary, so that the offer extracts the intermediary's entire collected demand. The collected demand function Ω^B extends (9) as follows:

$$\forall y > 0 : \Omega^B(y) = \sup_{\omega \leq y} qH(\omega)(\omega + \Omega^B(y-\omega)) + (1 - qH(\omega))q_b\Omega^B(y), \text{ and } \Omega^B(0) = 0. \quad (33)$$

The first part of the equation is the same as in (9), and the second part reflects the additional option of contacting an intermediary. Importantly, in this variant of the model, if q is low and q_b is large, we are likely to observe series of transactions with several intermediaries in a row. Since they all face the same financing needs, they all make the same offers, generating intermediation chains in which each dealer sells exactly the volume he bought from the previous dealer.

This extension delivers a prediction regarding the markets in which we expect to observe no-split chains. Indeed, such chains should be particularly prevalent when the distribution of customer demand is such that dealers have either zero (or very little) customer demand for the asset, or a demand large enough to absorb the entire initial volume. In such a situation, the entire volume is intermediated by a series of intermediaries with zero customer demand until a dealer with a large customer demand for the asset is located.

Offers to multiple dealers. We consider a variant of the model in which each dealer can make an offer successively to two other dealers, and is allowed to sell to both.[24] To analyze this game, we define the collected demand Ω_1 of a dealer with only one offer left, and the collected demand Ω_2 of a dealer with two offers left. One can show that these functions are defined as:

$$\forall y > 0 : \Omega_2(y) = \sup_{\omega,m,\ \omega \le m \le y} \Omega_1(y) + qH(\omega)[\omega + \Omega_2(m-\omega) + \Omega_1(y-m) - \Omega_1(y)] \quad (34)$$

$$\Omega_1(y) = \sup_{\omega \le y} qH(\omega)[\omega + \Omega_2(y-\omega)] \quad (35)$$

$$\Omega_1(0) = \Omega_2(0) = 0 \quad (36)$$

The definition of Ω_1 is similar to (9), but takes into account that the receiver of the offer will be able to make two offers, so that her expected transaction payoff is given by Ω_2. The interpretation of Ω_2 is that a dealer with financing needs y makes a first offer with $pv = m < y$ that does not exhaust his financing needs. Then, if the first offer is accepted, he behaves as a dealer with only one offer left and financing needs $y - m$, whereas if the offer is rejected his financing needs will still be y.

Instead of solving for a single function Ω as in the original model, in this extension one needs to solve for a system of two functional equations. The solution then gives all the equilibrium offers. The approach can be extended further to any number d of dealers that can be contacted, in which case one needs to solve for a system of d functional equations.[25]

Customer-to-dealer transactions. Our model focuses on interdealer transactions and

[24] The analysis would be more complicated in a common value environment, see Zhu (2012) for such a model of bargaining with multiple dealers.

[25] A simpler approach to analyze offers to multiple dealers is to assume that each dealer can contact d other dealers but only simultaneously, and cannot split the volume between them. Then the analysis of our baseline model goes through. The only difference is that the probability that an offer with target T is accepted is not $qH(\bar{\omega})$ but $1 - [(1-q) + q(1 - H(\bar{\omega}))]^d$.

does not include mark-ups charged to customers. However, we can introduce an endogenous customer-to-dealer transaction by endogenizing the initial offer (p_0, v_0). Let D_0 be a customer with a large endowment in the asset and a need for cash, so that getting y_0 is worth to him $U(y_0)$, with $U(0) = 0$, $U' \geq 0$, and $U'' \leq 0$. D_0 can either borrow y_0 at rate r, or make an offer (p_0, v_0) to dealer D_1. D_0 is exactly in the same position as any other D_n when making an offer, so that his expected payoff is $U(y_0) - ry_0 + r\Omega(y_0)$. If Ω is differentiable, the optimal y_0 satisfies $U'(y_0) = r[1 - \Omega'(y_0)]$. When U' is large enough the optimal y_0 is such that $P(y_0) < \rho$ and $\frac{\rho - P(y_0)}{\rho}$ can be interpreted as the mark-up charged by dealer D_1 when buying from the client. In the model this markup is purely a compensation for the expected financing costs of D_1.

6.2 Repo transactions and rehypothecation

In this section we use our framework to build a model of intermediation in the repurchase agreement ("repo") market. This is a natural alternative source of financing for dealers, who may choose to use the assets they bought as collateral to borrow cash from other dealers, instead of directly selling the assets to them. The setup is as described in Section 2.1, only with different endowments for dealers:

- Dealer D_1 accepted an offer (p_0, v_0) and can sell any number of assets to customers at price ρ. However, she cannot sell them until the "next day", whereas the amount $p_0 v_0$ has to be paid immediately. Thus, D_1 needs some bridge financing for one day. This can be done either by borrowing at rate r, or by receiving cash from D_2.

- Each dealer D_n for $n \geq 2$ has a cash endowment $\tilde{\omega}_n$, which follows a distribution G, and does not have customers herself. She can also borrow at rate r, or receive cash from D_{n+1}.

The dealers other than D_1 have no customers and hence no demand for the asset. However, they can lend their cash endowment and keep the asset as collateral. We assume that each dealer D_n can make a take-it-or-leave-it offer (m_n, i_n, c_n) to D_{n+1}. If the offer is accepted, then: (i) D_{n+1} immediately receives c_n units as collateral and gives m_n units of cash to D_n; (ii) D_n reimburses $m_n(1 + i_n)$ to D_{n+1} on the next day, and receives her collateral back. We assume that, for reasons outside of the model, repo contracts need to satisfy the following collateral constraint:

$$m_n \leq (1-h)\rho c_n, \tag{37}$$

which means that the maximal amount a dealer can borrow is lower than the fair value of the assets used as collateral, by a factor $1-h$. The exogenous parameter h corresponds to the haircut applied on the collateral. In addition, a dealer cannot pledge more units of collateral than he has received from the previous dealer.

Consider a dealer D_n, with $n \geq 2$, who receives an offer $(m_{n-1}, i_{n-1}, c_{n-1})$ satisfying (37). Denote $y_n = \max(m_{n-1} - \omega_n, 0)$. If he makes a new offer, D_n's payoff is:

$$m_{n-1}i_{n-1} - ry_n \quad \text{if } D_{n+1} \text{ rejects } (m_n, i_n, c_n) \tag{38}$$

$$m_{n-1}i_{n-1} - m_n i_n - r\max(m_{n-1} - \omega_n - m_n, 0). \quad \text{if } D_{n+1} \text{ accepts } (m_n, i_n, c_n). \tag{39}$$

These expressions are close to (3) and (4): the interest payment $m_{n-1}i_{n-1}$ plays the same role as intermediation revenues, and the amount m_{n-1} plays the same role as the value $p_{n-1}v_{n-1}$. Accordingly, we can build a rank-free equilibrium exactly in the same way, using the same function Ω defined by (9), where H corresponds here to the distribution of dealers' cash endowments. We obtain that D_n's equilibrium payoff and strategies are:

$$\pi^*(m_{n-1}, i_{n-1}, c_{n-1}, y_n) = m_{n-1}i_{n-1} - ry_n + r\Omega(y_n) \tag{40}$$

$$m_n = y_n \tag{41}$$

$$i_n = \frac{r[y_n - T(y_n) - \Omega(y_n - T(y_n))]}{y_n} \tag{42}$$

In particular, D_n can always make a new offer respecting the collateral constraint, so that this constraint plays no role in D_n's strategy for $n \geq 2$.

Consider now D_1. There are two cases. If $p_0v_0 \leq (1-h)\rho v_0$, then D_1 can choose $m_1 = p_0v_0$ while respecting the collateral constraint and $c_1 \leq v_0$. D_1's offer is then given by (42). If instead $p_0v_0 > (1-h)\rho v_0$, the collateral constraint binds. D_1 then borrows $m_1 = (1-h)\rho v_0$ on the repo market by pledging the entire v_0 as collateral, giving an expected benefit equal to $r\Omega((1-h)\rho v_0)$, and borrows the remaining $h\rho v_0$ on the unsecured market at a cost $rh\rho v_0$. The decision whether to accept (p_0, v_0) follows from computing the expected profit.

All our results on chain lengths in our benchmark model (Section 4) hold in this model. This is particularly interesting for this application as the number of dealers involved in a chain corresponds to the number of times the same asset serves as collateral for multiple dealers, a phenomenon known as *rehypothecation*.[26] In particular, the model predicts that the extent of rehypothecation is positively related to the financing needs of the first dealer (Proposition 2), and is lower under asymmetric information about dealers' endowments than under symmetric information (Proposition 4).

Moreover, this model delivers some insight on the impact of the haircut h. As long as $p_0 \leq (1-h)\rho$, the collateral constraint is slack and has no impact on the equilibrium. However, if the haircut increases enough to violate (37) then the collateral constraint binds. Higher haircuts then lead to lower borrowing and lower total interests paid.

Finally, like in the original model, we can reverse the trades by considering a dealer D_1 who sold the asset short and can buy it from new customers at a later date. The dealers D_n are endowed with the asset. D_1 can then borrow the units from D_2 by making a reverse repo offer, who can make an offer to D_3, and so on. The offers are determined by a value function, whose argument is not the financing needs but the quantity of assets a dealer needs to find.

7 Conclusion

Our paper proposes a framework to model the joint determination of prices and volumes in interdealer transactions and the structure of intermediation among dealers. We show that shocks to the customer demand for an asset, to the supply, and to intermediation frictions change both the number of dealers involved in transactions and the prices and quantities they trade, generating different correlations between prices and the number of dealers. Our modeling approach can be used in different settings. In addition to the extensions mentioned in Section 6, in future research it would be interesting to model the role of new forms of transaction on OTC markets (e.g., riskless principal, access to electronic platforms), or the dissemination of potentially toxic assets.[27]

[26] Other papers offer a more complete theoretical treatment of repo markets, see for instance Gottardi, Maurin, and Monnet (2019). Financial Stability Board (2017) reviews the associated policy concerns.

[27] See Di Maggio and Tahbaz-Alehi (2015) for a related model of interbank chains with moral hazard.

A Proofs

Proof of Property 1: Consider the first paragraph. D_n's profit is continuous with respect to ω and increasing. If the threshold is positive, it follows that the profit is null at the threshold.

Inequality (i): The stand-alone profit for an offer at a price less than or equal to $\frac{\rho}{1+r}$ is non-negative (from (5)), so D_n always accepts it even for $\omega = 0$.

Inequality (ii): Similarly, the stand-alone profit for an offer at a price less than ρ is positive for low enough financing needs.

Inequality (iii): Let D_n receive an offer at price ρ. Using (iv), which is proved below, any offer at a price strictly above ρ is rejected, so that D_n's expected transaction payoff is bounded above by qry_n (from the second line of (5)). Hence, if D_n receives an offer at price ρ, D_n's profit is bounded above by $-r(1-q)y_n$. As $q < 1$, this is negative unless $y_n = 0$, which shows $W(\rho, v) = \rho v$.

Inequality (iv): Consider D_n with financing needs y_n. Along the equilibrium, D_n 's expected transaction payoff divided by r, or *adjusted payoff*, is

$$E_n = \Phi(p_n, v_n)[r \min(y_n, p_n v_n) - (\rho - p_n)v_n], \tag{43}$$

where (p_n, v_n) is an optimal's offer for D_n. The proof relies on the following Lemma, which is proved in the Online Appendix:

Lemma A.1. *Consider a rank-free equilibrium. D_{n+1} targeted by D_n has lower financing needs and lower expected payoff than D_n: (i) $y_{n+1} \leq y_n$ and (ii) $E_{n+1} \leq E_n$.*

We now prove (iv). By contradiction, assume that a dealer D_m makes an offer (p_m, v_m) with $p_m > \rho$ and that this offer has a positive probability of being accepted. From the discussion of bubbles the Online Appendix, we know that the chain targeted by D_m has an infinite length, and that all targeted dealers make offers at prices above ρ. We consider this chain and index the dealers by n.

Lemma A.1 shows that the sequence of adjusted payoffs E_n is non-increasing. Since

$rE_n \leq q[ry_n - (\rho - p_n)v_n]$ (from the proof of Lemma A.1) we infer

$$rE_{n+1} \leq q[ry_n - (\rho - p_n)v_n]. \tag{44}$$

As D_{n+1}'s profit is non-negative, we have $(\rho - p_n)v_n - ry_{n+1} + rE_{n+1} \geq 0$. Using (44), this implies $(\rho - p_n)v_n - ry_{n+1} + q[ry_n - (\rho - p_n)v_n] \geq 0$, which is equivalent to

$$r(qy_n - y_{n+1}) \geq (1 - q)(p_n - \rho)v_n. \tag{45}$$

The sequence of financing needs y_n is non-increasing (Lemma A.1), hence it converges to a non-negative limit y_∞. The left-hand side of (45) thus converges to $r(q-1)y_\infty$, which is non-positive. On the right-hand side, the sequence $(p_n - \rho)v_n$ increases in n, so stays above $(p_m - \rho)v_m > 0$. Thus condition (45) cannot be met for all n, a contradiction. This concludes the proof. ∎

Proof of Property 2: See the text. ∎

Proof of Theorem 1: Consider the set $\mathcal{F}$ of bounded continuous functions β that satisfy $\beta(0) = 0$. Endow $\mathcal{F}$ with the sup-norm, denoted by $\|.\|_\infty$. Define the mapping Ψ on $\mathcal{F}$ by

$$\Psi(\beta)(y) = \max_{0 \leq \omega \leq y} qH(\omega)(\omega + \beta(y - \omega)). \tag{46}$$

The definition of Ψ is such that its fixed points Ω satisfy $\Omega(y) = \max_{0 \leq \omega \leq y} qH(\omega)(\omega + \Omega(y - \omega))$, hence coincide with the bounded and continuous solutions to (9). We show below that the function is well-defined, i.e., that the max in (46) is reached for each β in $\mathcal{F}$ and each y, and that Ψ maps $\mathcal{F}$ into $\mathcal{F}$. We then show that a fixed point exists and is unique by showing that Ψ satisfies Theorem 3.3 in Stokey and Lucas (1989), i.e., Ψ is a contraction.

Let us first prove that for each β in $\mathcal{F}$ and each y, the supremum in (46) is reached. There is no difficulty except if H has mass points, hence is discontinuous at those points. Consider a sequence ω_n converging to a discontinuity point ω of H such that the value of $H(\omega_n)(\omega_n + \beta(y - \omega_n))$ converges to the supremum. We show that this supremum is reached at ω. H jumps downward at ω. Thus for ω' higher than and close enough to ω, the value of $H(\omega')(\omega' + \beta(y - \omega'))$ is strictly lower than the value at ω. This implies that the sequence ω_n is lower than ω for n large enough. The supremum is thus reached at ω, by the left-continuity of H.

Let us now check that Ψ maps $\mathcal{F}$ into $\mathcal{F}$. Let β in $\mathcal{F}$. $\Psi(\beta)(0) = 0$ because $\beta(0) = 0$. For any $y > 0$, $\Psi(\beta)(y)$ is well defined since $0 \leq \omega \leq y$ is a compact set and β is continuous. $\Psi(\beta)$ is continuous on the set of positive y by the maximum principle, even if H is discontinuous by using a similar argument as above. Also $\Psi(\beta)(y)$ clearly tends to 0 with y, hence $\Psi(\beta)(y)$ is continuous at $y = 0$. $\Psi(\beta)$ is bounded: Since $H(\omega)\omega \leq \mathbb{E}(\tilde{\omega})$ (Markov's inequality) and $\mathbb{E}(\tilde{\omega})$ is finite, $H(\omega)\omega$ is bounded. It follows that

$$\|\Psi(\beta)\|_\infty \leq \max_{0 \leq \omega} qH(\omega)\omega + q\|\beta\|_\infty, \tag{47}$$

This proves that $\Psi(\beta)$ is in $\mathcal{F}$.

We now check that Blackwell's sufficient conditions - namely monotonicity and discounting - are satisfied. Working on (46), it is clear that $\beta_1 \geq \beta_2$ implies $\Psi(\beta_1) \geq \Psi(\beta_2)$ and $\Psi(\beta + a) \leq \Psi(\beta) + qa$ for any $a > 0$. This proves that Ψ is a contraction, and concludes the proof of the first part of Theorem 1.

We now prove the claimed properties on Ω.

Ω is non-decreasing in y: Due to the fact that Ψ is non-decreasing, Ω can be computed by iterate applications of Ψ starting at the null function $\beta = 0$. The argument is standard: Consider the sequence $\Omega_n = \Psi^n(0)$. The sequence is increasing because $\Omega_1 \geq \Omega_0 = 0$ and Ψ is non-decreasing. The sequence is bounded: Iteration of (47) implies that for each n:

$$\|\Omega_n\|_\infty \leq \frac{q}{1-q} \max_{0 \leq \omega \leq y} H(\omega)\omega.$$

Thus, the sequence Ω_n converges and the limit is a fixed point, which is necessarily equal to Ω. By induction, each element of the sequence $\Psi^n(0)$ is non-decreasing in y, hence the limit Ω as well.

Ω is bounded: This follows directly from the proof of the previous point.

Ω is lipshitz with constant q: It suffices to note that $\Psi(\beta)$ is q-lipshitz for any β in $\mathcal{F}$.

Since $\Omega(0) = 0$, this implies in particular that $\Omega(z) \leq qz$, a property we will use repeatedly.

To show that the target is bounded, it suffices to show that a supremum for $\omega H(\omega)$ is reached for ω smaller than some finite ω^{max}: since B and H are decreasing, the supremum of $H(\omega)(\omega + B(y - \omega))$ is necessarily reached in $[0, \omega^{max}]$ whatever y. The existence of ω^{max} is proved by showing $\lim_{t\to\infty} tH(t) = 0$. From Markov' inequality again applied to the truncated value of the distribution we have $tH(t) \leq \int_t^\infty \omega dG(\omega)$. Since the expected value of $\tilde{\omega}$ is finite, we have $\lim_{t\to\infty} \int_t^\infty \omega dG(\omega) = 0$. The result follows.

It remains to prove the last statement about the limit Ω^∞ of Ω when y tends to ∞. Since Ω is increasing and bounded, it surely converges to its upper bound: $\Omega^\infty = \|\Omega\|_\infty$. We prove that Ω^∞ satisfies (11): $\Omega^\infty = \max\limits_{0\leq\omega} qH(\omega)(\omega+\Omega^\infty)$. Bounding $\Omega(y-\omega)$ by Ω^∞ we immediately obtain that for each y

$$\Omega(y) \leq \max_{0\leq\omega\leq y} qH(\omega)(\omega + \Omega^\infty)$$

which implies

$$\Omega^\infty \leq \max_{0\leq\omega} qH(\omega)(\omega + \Omega^\infty).$$

We now prove the converse inequality. For each ω and $y \geq \omega$ we have

$$\Omega^\infty \geq \Omega(y) \geq qH(\omega)(\omega + \Omega(y - \omega))$$

Letting y increase indefinitely, the term on the right hand side converges to $qH(\omega)(\omega + \Omega^\infty)$ so we obtain that for each ω: $\Omega^\infty \geq qH(\omega)(\omega + \Omega^\infty)$ Since this is true for each ω, we obtain

$$\Omega^\infty \geq \max_{0\leq\omega} qH(\omega)(\omega + \Omega^\infty).$$

This proves the desired reverse inequality, hence $\Omega^\infty = \max\limits_{0\leq\omega} qH(\omega)(\omega + \Omega^\infty)$. ■

Proof of Theorem 2: The proof uses Lemma A.2 below, which is proved in the Online Appendix:

Lemma A.2. *Let f be a non decreasing $q-$lipshitz function defined over $\mathbb{R}^+$. Then, for ω less than pv and y:*

$$\min(pv, y) + f(pv - \omega) \leq pv + f(y - \omega).$$

We now prove the Theorem. Let T be a selection of $\mathcal{T}$.

1. We first check that the described strategies are feasible. Let D receive (p_0, v_0).[28] Refusing the offer is always feasible so we only have to consider the situation in which D accepts (p_0, v_0), which occurs if $(\rho - p_0)v_0 - ry + r\Omega(y) \geq 0$, and makes the offer $(P(y), V(y))$ described by (14). The offer is feasible if $0 \leq V(y) \leq v_0$. We prove the stronger property that $0 \leq V(y) \leq v_0 - v_1^C$. We have

$$\rho V(y) = (1 + r)y - r[T(y) + \Omega(y - T(y))].$$

$V(y) \geq 0$ since $y - T(y) - \Omega(y - T(y)) \geq 0$. Let us check $V(y) \leq v_0 - v_1^C$. For $y = 0$ the target is null, hence $V(y) = 0$. For $y > 0$, $V(y) \leq v_0 - v_1^C$ is equivalent to

$$\rho(v_0 - v_1^C) - (1 + r)y + r[T(y) + \Omega(y - T(y))] \geq 0. \tag{48}$$

Since $y = (p_0 v_0 - \rho v_1^C)$, replacing the value ρv_1^C by $p_0 v_0 - y$ we have $\rho(v_0 - v_1^C) - (1 + r)y = (\rho - p_0)v_0 - ry$. Furthermore, since $T(y)$ is in $\mathcal{T}(y)$, $\Omega(y) = qH(T(y))[T(y) + \Omega(y - T(y))]$, hence $\Omega(y) \leq T(y) + \Omega(y - T(y))$. We thus obtain

$$\rho(v_0 - v_1^C) - (1 + r)y + r[T(y) + \Omega(y - T(y))] \geq (\rho - p_0)v_0 - ry + r\Omega(y).$$

The right hand side of this inequality is non negative because the acceptance condition is satisfied. Hence (48) is satisfied: this implies $V(y) \leq v_0 - v_1^C$, the desired inequality.

2. Let us show that D's expected transaction payoff derived from a feasible offer (p, v), is not greater than $\Omega(y)$. Let us first characterize the acceptance probability of (p, v). According to the acceptance strategy, the receiver with customer demand ω accepts (p, v) if

$$(\rho - p)v - rz + r\Omega(z) \geq 0, \text{ where } z = \max(pv - \omega, 0). \tag{49}$$

If $p > \rho$, the offer is rejected for sure since $z - \Omega(z) \geq 0$. If $p \leq \rho$, a receiver with null financing needs accepts. Since $z - \Omega(z)$ is increasing, those who accept have financing needs

[28] For convenience, in this proof we use (p_0, v_0) to denote the offer received by any dealer D, not necessarily the offer received by D_1.

lower than the positive cut-off ξ that satisfies $(\rho - p)v - r\xi + r\Omega(\xi) = 0$. Equivalently, those who accept have customer demand larger than $\tau = pv - \xi$, where

$$(\rho - p)v - r(pv - \tau) + r\Omega(pv - \tau) = 0. \tag{50}$$

D's expected transaction payoff E from making the offer (p, v) is $qH(\tau)[r\min(pv, y) - (\rho - p)v]$, which, using (50), is also equal to

$$E = rqH(\tau)[\min(pv, y) - pv + \tau + \Omega(pv - \tau)]. \tag{51}$$

Consider two cases.

$\tau > y$. Then $H(\tau) \leq H(y)$. As $z - \Omega(z) \geq 0$ for each z, $pv - \tau - \Omega(pv - \tau) \geq 0$, which implies $[\min(pv, y) - pv + \tau + \Omega(pv - \tau)] \leq y$. Using this inequality and $H(\tau) \leq H(y)$ in (51) gives $E \leq rqH(y)y$; since $qH(y)y \leq \Omega(y)$, we obtain $E \leq r\Omega(y)$.

$\tau \leq y$. Since $\tau = pv - \xi$, the inequality $\tau \leq pv$ holds. We apply Lemma A.2 below to the q-lipshitz function Ω to obtain: $\min(pv, y) + \Omega(pv - \tau) \leq pv + \Omega(y - \tau)$. It follows that $[\min(pv, y) - pv + \tau + \Omega(pv - \tau)] \leq \tau + \Omega(y - \tau)$. Using this inequality in (51) gives

$$E \leq rqH(\tau)[\tau + \Omega(y - \tau)]. \tag{52}$$

Since $\tau \leq y$, the right hand side is lower than $r\Omega(y)$ by definition of Ω, so that $E \leq r\Omega(y)$.

We thus conclude that in each case $E \leq r\Omega(y)$: no offer (p, v) yields a benefit higher than $r\Omega(y)$.

3. Optimality of the strategies. Point 2 implies that D accepting the offer (p_0, v_0) will obtain a profit at most equal to $(\rho - p_0)v_0 - ry + r\Omega(y)$. Consider two cases.

a. Let $(\rho - p_0)v_0 - ry + r\Omega(y) < 0$. The profit is surely negative by accepting (p_0, v_0), hence it is optimal for D to refuse (p_0, v_0), i.e., $A(p_0, v_0, \omega) = 0$ is optimal.

b. Let $(\rho - p_0)v_0 - ry + r\Omega(y) \geq 0$. We check that the offer $(P(y), V(y))$ achieves the payoff $r\Omega(y)$. This will prove that such an offer is optimal and furthermore that D's profit is equal to $(\rho - p_0)v_0 - ry + r\Omega(y)$. As this quantity is non-negative by assumption, it is optimal for D to accept (p_0, v_0) i.e. $A(p_0, v_0, \omega) = 1$ is optimal.

Let us show that the offer $(P(y), V(y))$ achieves $r\Omega(y)$. By point 1, we know that $(P(y), V(y))$ is feasible. We prove $\tau = T(y)$ is a threshold associated to $(P(y), V(y))$. Such a threshold τ is defined by (50), which using $y = P(y)V(y)$ by the first equation in (14), writes $(\rho - P(y))V(y) - r(y - \tau) + r\Omega(y - \tau) = 0$. The second equation in (14) yields that $T(y)$ is a threshold.

At $(P(y), V(y))$, D's payoff is given by (51) where $\tau = T(y)$ and $\min(pv, y) - pv$ is null. This gives

$$E = rqH(T(y))(T(y) + \Omega(y - T(y)),$$

E is thus equal to $r\Omega(y)$ since $T(y)$ belongs to $\mathcal{T}(y)$. This ends the proof of the Theorem. ■

Proof of Proposition 1: 1. The results are straightforward from the analytical expressions for $P(y)$ and $V(y)$, and the asymptotic results have been proved in the text.

2. The expression (18) for the receiver's profit follows from the fact that $\pi^*(P(y), V(y), z) = (\rho - P(y))V(y) - rz + r\Omega(z)$. Using that this profit is null at $\omega = T(y)$ (or equivalently the second equation in (14) gives the expression. It is obviously decreasing in z. The inventory is decreasing in z since the volume $V(z)$ is increasing in z. ■

Proof of Lemma 1: 1. *Z is non-decreasing.* The maximization problem (9) can be stated equivalently in terms of the target's financing needs $z = y - \omega$, as maximizing $F(z, y) = H(y - z)[y - z + \Omega(z)]$ with respect to $z, 0 \leq z \leq y$. Denoting the maximizers by ζ, one has $\zeta(y) = y - T(y)$. The monotonicity of ζ follows from the (ordinal) single-crossing property satisfied by F:

$$F(z, y) - F(z', y) \geq 0 \text{ for } y \geq z > z' \Rightarrow F(z, y') - F(z', y') > 0 \text{ for } y' > y. \tag{53}$$

Indeed, suppose (53). Let $z \in \zeta(y)$; then $z \leq y$ and $F(z, y) - F(z', y) \geq 0$ for any $z' \leq y$, in particular for any $z' < z$. Hence (53) implies $F(z, y') - F(z', y') > 0$ for any $y' > y$ and $z' < z$: z' is surely not in $\zeta(y')$.

It remains to prove that F satisfies (53). Observe that

$$F(z,y)-F(z',y)\geq 0 \Leftrightarrow \frac{y-z+\Omega(z)}{y-z'+\Omega(z')}-\frac{H(y-z')}{H(y-z)}\geq 0.$$

Thus it suffices to show that the function on the right hand side is increasing in y. The first term writes as

$$\frac{y-z+\Omega(z)}{y-z'+\Omega(z')}=1+\frac{z'-\Omega(z')-z+\Omega(z)}{y-z'+\Omega(z')},$$

hence is increasing in y because $z'-\Omega(z')<z-\Omega(z)$ for $z'<z$ (because Ω is $q-$lipshitz, Theorem 1). The second term is nondecreasing in y by the log-concavity of H (Assumption (A1)): The log derivative w.r.t. y of $-\frac{H(y-z')}{H(y-z)}$ is $-\frac{H'}{H}(y-z')+\frac{H'}{H}(y-z)$, which is non-negative since $z'<z$. This proves (53).

2. *There is a value $\underline{y}_1$ such that $T(y)=\{y\}$ and $Z(y)=\{0\}$ for $y<\underline{y}_1$ and positive for $y<\underline{y}_1$.* We show that, under Assumption (A1), there is a value $\underline{y}>0$ such that $\Omega(y)=qH(y)y$ for $y<\underline{y}$ and that the target is unique given by $\omega=y$, hence $Z(y)=\{0\}$. Since Z is non-decreasing and targets have an upper-bound (by Theorem), $Z(y)$ is strictly positive for y larger than this upper-bound. Hence it suffices to define $\underline{y}_1$ as the supremum value such that $0\in Z(y)$.

Consider the recursive equation (54):

$$\forall y>0, \Omega(y)=\sup_{\omega\leq y} qH(\omega)(\omega+\Omega(y-\omega)),\text{ and } \Omega(0)=0. \tag{54}$$

Choosing $\omega=y$ implies $\Omega(y)\geq qH(y)y$. We show that, under Assumption (A1), there is a value $\underline{y}>0$ such that $\Omega(y)=qH(y)y$ for $y<\underline{y}$ and that the target is unique given by $\omega=y$.

First note that since $\Omega(z)\leq qz$ for any z, we infer from the recursive equation (9) that $\Omega(y)\leq\max_{\omega\leq y} qH(\omega)(\omega+q(y-\omega))$. Denoting by $\delta(\omega,y)$ the difference between $H(y)y$ and $H(\omega)(\omega+q(y-\omega))$, it thus suffices to show that there is a value $\underline{y}>0$ such that $\delta(\omega,y)>0$ for any ω and y that satisfies $\omega<y$ and $y<\underline{y}$. $\delta(\omega,y)$ writes

$$\delta(\omega,y)=[H(y)-H(\omega)]y+(1-q)H(\omega)(y-\omega).$$

Since G admits a continuous density, H is liphchitz on any bounded interval. Thus, there is k such that for $\omega \leq y \leq 1$: $H(y) - H(\omega) \geq -k(y - \omega)$, which implies

$$\omega \leq y \leq 1 : \delta(\omega, y) \geq [-ky + (1-q)H(y)](y - \omega).$$

For y small enough the term inside the square brackets is strictly positive because the distribution is not concentrated on zero. We thus obtain the desired result.[29]

3. *There is a positive lower bound ω_{min} to any target of a dealer D who have financing needs $y > \underline{y}_1$.* Let $\omega \in \mathcal{T}(y)$. $\Omega(y) = qH(\omega)(\omega + \Omega(y - \omega))$. This equation implies $\Omega(y) \leq \frac{q}{1-q}\omega$ since $\Omega(y) \geq \Omega(y - \omega)$ and H is less than 1. Hence, using $\Omega(y) \geq \Omega(\underline{y}_1) = qH(\underline{y}_1)\underline{y}_1$ (since Ω is increasing) we obtain

$$\text{for } y > \underline{y}_1 \text{ and } \omega \in \mathcal{T}(y) : \omega \geq \omega_{min} = (1-q)H(\underline{y}_1)\underline{y}_1,$$

which concludes the proof. ∎

Proof of Corollary 1: See the text. ∎

Proof of Proposition 2: Lemma 1 directly implies that the targeted length $N_T(y)$ is finite and non-decreasing in y. The fact that dissemination length is always lower than targeted lengths can be shown by induction on the targeted length. If $N_T(y) = 2$, then the offer is at price ρ and the receiver does not make a new offer. Hence, the dissemination length 2 (if the offer is accepted) or 1 (if it is rejected). Assume that dissemination lengths are lower than targeted lengths for targeted lengths up to N. Consider a targeted length of size $N+1$, starting with a dealer D_1 with financing needs y_1. If $\omega_2 < T(y_1)$, D_1's offer is rejected and the dissemination length is $1 < N + 1$. Otherwise, D_2 has financing needs $y_1 - \omega_2 \leq y_1 - T(y_1)$. Hence, $N_T(y_2) \leq N_T(y_1 - T(y_1)) = N_T(y_1) - 1 = N$. Using the induction assumption, the dissemination length starting with D_2 is lower than N, hence the length starting with D_1 is less than $N + 1$. ∎

[29]Note that we can infer that there are two targets at $\underline{y}_1$: $\underline{y}_1$ and a strictly lower value.

Proof of Lemma 2: We will work with a weaker assumption than (A2):

$$G \text{ is such that } (11): \Omega^{\infty} = \max_{\omega} qH(\omega)(\omega + \Omega^{\infty}), \text{ has a unique maximizer } \omega^{\infty}. \tag{A2}$$

If G admits a density and satisfies (A1) then (A2) is satisfied. If G is discrete with support $(\omega_k)_{1\leq k\leq K}$, where $\omega_k < \omega_{k+1}$, then if the sequence $\omega_k H(\omega_k)$ increases up to a unique value k^* and then decreases, (A2) is satisfied.

Define the function F: $x \mapsto \max_{\omega} H(\omega)(\omega + x)$ and let denote $t(x)$ the set of maximizers. The fact that the maximum is reached even if H is discontinuous (on the right) follows the same argument as in Theorem 1. We know from Theorem 1 that Ω_∞ satisfies $qF(\Omega_\infty) = \Omega_\infty$. We show in step 1 that Ω_∞ is the unique solution to that equation and in step 2 that $\mathcal{T}(y)$ converges to those in $t(\Omega_\infty)$. Since (A2) implies that $t(\Omega_\infty)$ is unique, denoted ω_∞), this implies the convergence of $\mathcal{T}(y)$ to ω_∞) when y increases.

Step 1. Let two values x and x'. By definition, we have for $\tau \in t(x)$ and $\tau' \in t(x')$

$$F(x) = H(\tau)(\tau + x) \geq H(\tau')(\tau' + x) \text{ and } F(x') = H(\tau')(\tau' + x') \geq H(\tau)(\tau + x').$$

We derive:

$$H(\tau)(x - x') \geq F(x) - F(x') \geq H(\tau')(x - x').$$

Let $x > x'$. These inequalities imply $H(\tau)(x - x') \geq F(x) - F(x') > 0$ and $H(\tau) > H(\tau')$. Hence $\tau < \tau'$ and the function F is 1-lipshitz. This implies that the solution Ω_∞ to the equation $qF(x) = x$ is the unique one. Furthermore for any $\tau \in t(x)$, $\tau \leq t(0)$, where $t(0)$ is a maximizer of $H(\omega)\omega$.

Step 2. When y increases, the elements in $\mathcal{T}(y)$ converge to those in $t(\Omega_\infty)$. Consider two elements τ and t. We have

$$H(t)(t + \Omega(y - t)) - H(\tau)(\tau + \Omega(y - \tau) \;=\; H(t)(t + \Omega_\infty) - H(\tau)(\tau + \Omega_\infty) \tag{55}$$

$$-H(t)(\Omega_\infty - \Omega(y - t)) \;+\; H(\tau)[\Omega_\infty - \Omega(y - \tau)] \tag{56}$$

Take $\tau \notin t(\Omega_\infty)$ and $t \in t(\Omega_\infty)$. By definition, the expression on the RHS of (55) is strictly positive. Consider (56). Since $\tau \leq t(0)$ and $\lim_{z\to\infty} \Omega(z) = \Omega_\infty$, the expression in (56) converges to zero when y goes to infinity. This implies that $H(t)(t+\Omega(y-t))-H(\tau)(\tau+\Omega(y-\tau) > 0$ for y large enough: $\tau \notin \mathcal{T}(y)$ for y large enough. ∎

Proof of Proposition 3: The result on dissemination length is proved in the text. For the customer purchases, from the Online Appendix we have $C_F(p_0v_0, \omega_1, \sigma) = \frac{1}{\rho}\min\left(p_0v_0, \sum_{j=1}^{M}\omega_j\right)$ when D_1 accepts the offer. We will show that $C_R(p_0v_0, \omega_1, \sigma)$ is lower than this quantity. Note that we have the financing needs equations $y_n = y_{n-1} - \omega_n$ as long as $n < m = N_R(p_0v_0 - \omega_1, \sigma)$. At the last step m, two cases can occur.

Case 1: The chain stops with $y_m = 0$, hence m sells to her customers x such that $\rho x = y_{m-1}$. Adding all the financing needs equations, we obtain $\rho\sum_{j=1}^{m-1} v_j^C + \rho x = p_0v_0$, with $p_0v_0 \leq \sum_{j=1}^{M}\omega_j$. We obtain $C_R(p_0v_0, \omega_1, \sigma) = C_F(p_0v_0, \omega_1, \sigma)$.

Case 2: The chain stops with $y_m > 0$: D_m's offer is rejected and $m \leq N_F(p_0v_0 - \omega_1, \sigma)$. Adding all the financing needs equations, we obtain $\sum_{j=1}^{m}\omega_j = p_0v_0 - y_m$. As $m \leq M$ the customer purchases are lower than both p_0v_0 and $\sum_{j=1}^{M}\omega_j$, hence $C_R(p_0v_0, \omega_1, \sigma) \leq C_F(p_0v_0, \omega_1, \sigma)$. ∎

Proof of Proposition 4: Let us prove that $\lim\limits_{y\to+\infty} \mathbb{E}(N_R(y, .)) = \ell_{\omega^\infty}$. Using (24), $\mathbb{E}(N_R(y, .))$ is bounded below by 1, and bounded above by $\ell_0 = \frac{1}{1-q}$ (the expected length of a sequence that continues until one dealer is inactive). Hence, $\mathbb{E}(N_R(y, .))$ admits a finite superior limit $\overline{\ell}$, and a finite inferior limit $\underline{\ell}$.

First, we show that $\underline{\ell} \geq \ell_{\omega^\infty}$. Let $\ell < \underline{\ell}$. By definition of $\underline{\ell}$, there is z such that $\mathbb{E}(N_R(z, \sigma)) \geq \ell$. Consider $y > z + \omega^*$. As $T(y) \leq \omega^*$, this implies that $y - z > T(y)$, and we can write:

$$\int_{T(y)}^{y} \mathbb{E}(N_R(y-\omega, \sigma))dG(\omega) \geq \int_{T(y)}^{y-z} \ell dG(\omega) \geq [G(y-z) - G(T(y))]\ell.$$

Plugging this inequality into (24), we obtain: For any $y > z$, $\mathbb{E}(N_R(y, .)) \geq 1 + qH(y) + q[G(y-z) - G(T(y))]\ell$. Taking the limit in y yields $\underline{\ell} \geq 1 + q\ell H(\omega^\infty)$. Since this is true for any $\ell < \underline{\ell}$, we obtain $\underline{\ell} \geq 1 + q\underline{\ell}H(\omega^\infty)$, which is equivalent to $\underline{\ell} \geq \ell_{\omega^\infty}$.

Second, we show that $\overline{\ell} \geq \ell_{\omega^\infty}$. By definition $\mathbb{E}(N_R(z,\sigma)) \leq \overline{\ell}$ for any z. Using this inequality to bound the integral in (24), we obtain $\mathbb{E}(N_R(y,.)) \leq 1 + qH(y) + q\overline{\ell}[G(y) - G(T(y))]$. Taking the limit in y yields $\overline{\ell} \leq 1 + q\overline{\ell}H(\omega^\infty)$, which is equivalent to $\overline{\ell} \leq \ell_{\omega^\infty}$.

We thus have $\ell_{\omega^\infty} \geq \overline{\ell} \geq \underline{\ell} \geq \ell_{\omega^\infty}$, which implies that $\mathbb{E}(N_R(y,.))$ converges to ℓ_{ω^∞}.

Since $\mathbb{E}(\widehat{N}_R(y,.))$ is given by a similar expression as $\mathbb{E}(N_R(y,.))$ with a degenerate distribution, the same argument shows that $\lim_{y\to+\infty} \mathbb{E}(N_F(y,.)) = \ell_{\bar{\omega}}$. Since the distribution is degenerate, $H(\bar{\omega}) = 1$ and hence $\ell_{\bar{\omega}} = \ell_0$.

Finally, the asymptotic behavior of $N_T(y)$ follows from the fact that $T(y)$ converges to ω^∞ when y is large, so that the targeted length increases by one unit for every increment of y of length ω^∞. As a result, $N_T(y)$ goes to infinity and is asymptotically equivalent to $\frac{y}{\omega^\infty}$. The same reasoning applies to $\widehat{N}_T(y)$, as $T(y) = \bar{\omega}$ for any y. Note that the proposition also holds under (A2). ■

Proofs of Implications: All the implications are immediately deduced from previous results or are proved in the main text. The only exception is Implication 2: we need to prive that v_n^I decreases after a fall in r for $n > 1$. Using (19), D_n's inventory varies with r as $V(y_{n-1}) - V(y_n)$, i.e., using (16), as $r[(Z(y_{n-1}) - \Omega(Z(y_{n-1})) - (Z(y_n) - \Omega(Z(y_n))]$. The term within the square brackets is positive under Assumption (A1): (i) $Z(.)$ is an increasing function under (A1) and $y_{n-1} \geq y_n$, so that $Z(y_{n-1}) \geq Z(y_n)$; (ii) $z - \Omega(z)$ is an increasing function. Hence, D_n's inventory is increasing in r. ■

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