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The Need for Fees at a DEX: How Increases in Fees Can Increase DEX Trading Volume

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Abstract

We demonstrate that increasing trading fees at a decentralized exchange (DEX) can increase DEX trading volume. This result arises due to the fact that higher DEX fees can endogenously reduce the price impact of trading at the DEX, thereby reducing the overall DEX trading cost and driving trading activity to the DEX from competing exchanges. The referenced relationship between fees and price impacts arises because DEXs employ a mechanical pricing rule whereby price impacts reduce with the DEX inventory level, and DEXs acquire inventory by offering DEX fee revenue in exchange for capital from investors used to finance the DEX inventory. When fees are sufficiently low, increases in the DEX fee level lead to higher DEX fee revenue and higher DEX investment returns, thereby increasing DEX inventory; in turn, price impacts decline and so too do overall trading costs, resulting in an increase in DEX trading volume.

Keywords: Decentralized Exchange, DEX, Automated Market Makers, AMM

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1 Introduction

We provide an economic model of a decentralized exchange (DEX) with the aim of understanding the role that DEX fees play in the adoption of the DEX as a trading platform. Our main finding is that an increase in DEX trading fees can increase the equilibrium DEX trading volume and therefore the use of the DEX. This result is particularly noteworthy because it highlights an important distinction between a DEX and a centralized exchange (CEX); more precisely, an increase in CEX trading fees would increase CEX trading costs and thereby unambiguously reduce CEX trading volume. In contrast, our results highlight that such an unambiguously negative relationship between trading fees and trading volume does not arise for a DEX.

We derive our main finding, that increases in DEX trading fees can increase DEX trading volume, by showing that an increase in DEX trading fees can reduce overall DEX trading costs, thereby driving trading activity from competing exchanges to the DEX. In order to understand why this result holds, first note that DEXs employ a mechanical pricing rule which imposes an automatic price impact on any trade with the DEX. Therefore, the cost of trading with the DEX not only arises from the trading fee, but also from the price impact of the trade. We then show that due to the form of the mechanical pricing rule, the price impact of DEX trading is strictly decreasing in the DEX inventory level which is the amount of funds provided by outside investors that are used as liquidity to facilitate trades. Therefore, whenever increases in the DEX fee level lead to increases in the DEX inventory then the traders will always be guaranteed a lower price impact for trading with the DEX. Importantly, DEXs acquire inventory by offering a pro-rata share of DEX trading fees to the investors who provide such inventory. This implies that, in equilibrium, an increase in the DEX fee level can result in an increase in the overall DEX fee revenue, increasing the investor return from financing DEX inventory. Whenever this is the case, an increase in DEX fees will lead to a higher level of equilibrium DEX inventory, lower price impacts, and therefore higher equilibrium DEX trading volume.

Formally, we model a one-period setting consisting of two types of agents, investors and traders, and two types of trading exchanges, a DEX and a CEX. A unit measure of investors arrive at the beginning of the period, each who possess a unit of capital which they decide to invest either in the DEX (i.e. to provide inventory) or to invest in an alternative investment opportunity which generates an exogenous and known return. Subsequently, liquidity traders with heterogeneous demand arrive according to a Poisson Process and trade at either the DEX or the CEX, selecting whichever exchange offers the lower trading cost. Finally, at the end of the period, each investor who invested in the DEX receives a pro-rata share of all DEX trading fees, whereas each investor who invested in her alternative investment opportunity receives their known exogenous return.

Our model entails two sources of trading costs: trading fees and price impacts. Trading fees arise because we assume that each exchange charges an exogenous proportional fee on any volume traded, whereas price impacts arise whenever the size of a trade affects the execution price of that trade. To model the CEX we follow Glosten and Milgrom (1985) while abstracting from asymmetric information, implying that the CEX always offers execution at fair value (i.e., no price impact). In contrast, as per John et al. (2022a), we assume that the DEX employs a mechanical pricing rule so that the execution price at the DEX approaches fair value for arbitrarily small trade sizes but diverges from fair value as the trade size diverges from zero (i.e., non-zero price impact).

Our main result, Proposition 3.1, establishes that an increase in the DEX fee level generates an increase in the DEX trading volume so long as the initial DEX fee level is sufficiently small. This result arises because the referenced increase in the DEX fee level reduces the price impact of DEX trading to such an extent that the overall DEX trading cost falls for the marginal trader who would be indifferent between the DEX and the CEX in the absence of a DEX fee level increase. In turn, since traders select between the DEX and the CEX on the basis of whichever provides the lower trading cost, the referenced increase in the DEX fee level drives traders to the DEX from the CEX after which the marginal trader will have a strictly higher volume trade demand, implying an increase in the equilibrium DEX trading volume. We establish the described channel that generates our main result via Propositions 3.2 and 3.3. More specifically, Proposition 3.2 establishes that an increase in the DEX fee level reduces trading costs for marginal traders so long as the initial DEX fee level is sufficiently small, whereas Proposition 3.3 establishes that an increase in the DEX fee level reduces the price impact of DEX trading so long as the initial DEX fee level is sufficiently small.

The result of Proposition 3.3, that an increase in the DEX fee level reduces the DEX price impact when the DEX fee level is initially sufficiently small, arises because the DEX's mechanical pricing rule embeds a negative relationship between the DEX price impact and the DEX inventory level and the DEX inventory level is increasing in the DEX fee when it is sufficiently small. More explicitly, the DEX acquires inventory by offering a pro-rata share of DEX trading fee revenues to investors in exchange for financing the capital which is used as DEX inventory. Therefore, an increase in the DEX fee level can increase overall DEX fee revenues and thus the return on investment from financing the DEX inventory. Then, given that investors select the investment that generates the highest expected return, the referenced increase in the DEX fee level endogenously increases DEX investment which increases DEX inventory and, as mentioned above, reduces the price impact of DEX trading. We formally establish the referenced relationships in Propositions 3.4 - 3.6. In particular, we demonstrate that increases in the DEX investment level monotonically reduce the price impact of DEX trading in Proposition 3.4, whereas we establish that, for a sufficiently small initial DEX fee level, an increase in the DEX fee level increases the DEX investment return and also the DEX investment level in Propositions 3.5 and 3.6.

Our paper contributes to the literature on the economics of blockchain and cryptoassets. John et al. (2022b) and John et al. (2022a) provide surveys of that literature. Some notable papers examining the economics of blockchain include Biais et al. (2019), Easley et al. (2019), Makarov and Schoar (2019), Huberman et al. (2021), Saleh (2021) and Biais et al. (2022). Within the economics of blockchain literature, our paper specifically contributes to the literature that examines Decentralized Finance (DeFi) applications and DEXs in particular. Prominent papers that examine DeFi blockchains include Cong et al. (2021), Cong et al. (2022) and Mayer (2022), whereas prominent papers studying DEXs include Barbon and Ranaldo (2021), Capponi and Jia (2021), Lehar and Parlour (2021) and Park (2021). Our paper differs from others papers that examine DEXs in that we focus upon the theoretical implications of varying DEX fee levels.

2 Model

We model a setting in which time is indexed by $t \in [0, 1]$ and in which there are two types of agents: investors and traders. A unit measure of investors arrive at $t = 0$ and each possesses a unit of capital. Upon arrival, each investor decides whether to provide her capital to a decentralized exchange (DEX) or to invest in an alternative investment instead. Thereafter, from $t = 0$ to $t = 1$, traders arrive sequentially and decide whether to trade at the DEX or to trade at a centralized exchange (CEX). At $t = 1$, all investors realize pay-offs.

2.1 Exchanges

We model two exchanges, one CEX and one DEX, each of which offers trading of a single cryptoasset against a USD-equivalent token. Hereafter, for exposition, we refer to the single asset as ETH and to the USD-equivalent token as USD.[1]

In general, trading ETH entails two costs: a direct cost arising from the price of the ETH and an indirect cost arising from fees charged on the trading of ETH. More explicitly, we denote by $P_i(\delta) \geqslant 0$ the per unit ETH price (in USD) for a trade of δ ETH at exchange i (defined precisely below) so that a trade of $\delta \in \mathbb{R}$ ETH entails a direct cost of $\delta \times P_i(\delta)$ USD at exchange $i \in \{CEX, DEX\}$. When $\delta > 0$ the trade corresponds to a buy of $|\delta|$ ETH, whereas when $\delta < 0$ the trade refers to a sell of $|\delta|$ ETH; note that a sale of ETH entails a negative direct cost (i.e., $\delta \times P_i(\delta) < 0$ whenever $\delta < 0$) because the sale generates proceeds for the seller rather than an expense. In addition to the referenced direct cost, trading with an exchange also entails an indirect cost arising from the trading fee charged by the exchange which is proportional to the size of the trade. More formally, we denote by $V \geqslant 0$ the fair value of ETH (in USD) and by $f_i \in [0, \overline{f}]$ the proportional fee charged at exchange $i \in$ {CEX,DEX}.[2] Then, a trade of δ ETH entails a proportional fee of $f_i \times |\delta|$ (denominated in ETH) at exchange i so that the overall fee for a trade of δ ETH equals $f_i \times |\delta| \times V$ in USD. Formally, the overall cost of trading δ ETH at exchange i, which we denote by $\Psi_i(\delta)$, is given as follows:

$$\underbrace{\Psi_i(\delta)}_{\text{Total Trading Cost}} = \underbrace{P_i(\delta) \times \delta}_{\text{Execution Price Cost}} + \underbrace{f_i \times |\delta| \times V}_{\text{Trading Fees Cost}} \tag{1}$$

The key difference between a CEX and a DEX arises in the specification of the execution price (i.e., $P_i(\delta)$). Specifically, we assume that market-makers at the CEX price ETH according to Glosten and Milgrom (1985). Our model abstracts from asymmetric information which thereby implies that the CEX always prices ETH at its known fair value V:

[1] Formally, the reader should consider the trading as being ETH against a stablecoin pegged to USD. Such pairs (e.g., ETH-USDC, ETH-USDT) are offered by both centralized and decentralized exchanges.

[2] We assume $\overline{f} \leqslant 25\%$ and $V \geqslant \frac{1}{2}$. These assumptions simplify our equilibrium solution but are not necessary.

$$P_{CEX}(\delta) = V \tag{2}$$

In contrast, ETH pricing at the DEX is mechanical and determined according to an exogenous function known as an Automated Market Maker (AMM) function. The referenced mechanical function determines a price as a function of not only the trade size, δ, but also the DEX inventory levels for ETH and USD. We assume the DEX employs the most common AMM function used in practice, the Constant Product Automated Market Maker (CPAMM) function, which implies the following pricing function (see John et al. 2022a):

$$P_{DEX}(\delta) := \Xi(I_{USD}, I_{ETH}, \delta) \equiv \begin{cases} \frac{I_{USD}}{I_{ETH}-\delta} & \text{if } \delta < I_{ETH} \\ \infty & \text{if } \delta \geqslant I_{ETH} \end{cases} \tag{3}$$

where I_{USD} and I_{ETH} denote the USD and ETH inventory levels at the DEX respectively and $\Xi(I_{USD}, I_{ETH}, \delta)$ is the functional form of the CPAMM pricing function. We follow the specification of UniSwap V2 and require that investors who provide liquidity do so by adding both ETH and USD to the inventory in a fixed proportion (see John et al. 2022a for details). Moreover, we ensure the absence of arbitrage across the DEX and CEX by requiring that this fixed proportion is such that the marginal ETH price at the DEX is initially aligned with the ETH price at the CEX:

$$\lim_{\delta \to 0^+} P_{DEX}(\delta) = V = P_{CEX}(\delta) \tag{4}$$

2.2 Traders

We assume that there exist two types of traders: liquidity traders and opportunistic traders. Liquidity traders possess an exogenous trading demand and must satisfy their demand either at the CEX or at the DEX. In contrast, opportunistic traders exploit the trading of liquidity traders. In more detail, the DEX pricing function (see Equation 3) mechanically implies that the ETH price at the DEX moves in the direction of a trade (i.e., a buy increases ETH prices, whereas a sell decreases ETH prices) so that, even though DEX and CEX marginal prices are initially aligned, a liquidity trade in one direction produces an opportunity to trade in the opposite direction at a price which is favorable relative to fair value. We assume that such opportunities are seized upon immediately so that any movement in the marginal ETH price at the DEX away from fair value is subsequently traded away by traders who wait opportunistically for such price movements before executing their trade to benefit from lower trading costs. We refer to such traders that seize favorable trading opportunities as opportunistic traders, and note that such traders can be interpreted as a type of liquidity trader that is sufficiently time insensitive so that they can wait for the price to move in a favorable direction before executing a trade without incurring a large opportunity cost.

Formally, we assume liquidity traders arrive randomly over time $t \in [0,1]$ according to a Poisson Process with unit intensity. We index liquidity traders by $j \in \{1, ..., N\}$ where $N \sim$ Poisson(1) represents the random number of arriving liquidity traders. Liquidity Trader j possesses trading demand $\delta_j \sim U[-1,1]$ where $\delta_j > 0$ represents the need to buy $|\delta_j|$ ETH, while $\delta_j < 0$ represents the need to sell $|\delta_j|$ ETH.

Each liquidity trader decides whether to trade at the DEX or at the CEX by selecting the exchange that minimizes her cost of trading as follows:

$$i(j) = \underset{i \in \{CEX, DEX\}}{\arg\min} \; \Psi_i(\delta_j) \tag{5}$$

In turn, we define D as the random set of liquidity traders who optimally trade at the DEX, stated explicitly as follows:

$$D = \{j : i(j) = DEX\} \tag{6}$$

As discussed, we assume that opportunistic traders arrive immediately after liquidity traders to seize favorable trading opportunities until their trading realigns the marginal DEX price with the CEX price which is equal to the ETH fair value. Due to the mechanical pricing rule at a DEX, this re-alignment occurs only after the DEX has experienced trading of an equal magnitude but opposite direction as the liquidity trade that generated the opportunity. Whether such trading occurs across multiple opportunistic trades or a single opportunistic trade is without loss of generality, so we assume that the re-alignment occurs through a single trade for exposition; more formally, we assume that every liquidity trade of size δ_j made at the DEX is instantly reversed by an equivalent opportunistic trade of size $-\delta_j$.

2.3 Investors

We assume that there exists a unit measure of investors indexed by $k \in [0,1]$. Each Investor k possesses a unit of capital and may invest that capital in either the DEX or an alternative investment. We assume that the alternative investment for Investor k provides her a net expected return of $\rho_k \sim G[0,1]$ where $G(\rho) = \rho^{\frac{1}{\theta}}$ with $\theta > 1$ denotes the cumulative distribution function of investor returns. In contrast, investing in the DEX entitles an investor to an endogenous expected return equal to the pro-rata share of fees generated by the DEX. We denote the expected return from investing in the DEX by r_{DEX}, which is given explicitly as follows:[3]

$$r_{DEX} = \frac{\text{Total Expected Fees}}{\text{Total Invested Capital}} = \frac{2 \times \mathbb{E}[\sum_{j:j \in D} f_{DEX} \times |\delta_j| \times V]}{I} \tag{7}$$

where I corresponds to the total DEX investment (in USD) and $\mathbb{E}[\sum_{j:j \in D} f_{DEX} \times |\delta_j| \times V]$ corresponds to the expected fee revenue from liquidity traders. The multiplicative factor of 2 in the numerator of Equation (7) reflects the fact that any fee paid by a liquidity trader at the DEX is duplicated by fees from opportunistic trading that returns the DEX price back to its fundamental value; all our results hold even without this factor of 2.

We assume that each investor is risk neutral and therefore invests in the investment opportunity that provides her the highest expected return. Consequently, Investor k invests in the DEX if and only if $r_{DEX} \geqslant \rho_k$. Therefore, the equilibrium DEX investment must satisfy the following equation:

$$I = G(r_{DEX}) = (r_{DEX})^{\frac{1}{\theta}} \tag{8}$$

2.4 Model Solution

Formally, an equilibrium is a DEX investment return $r^{\star}_{DEX}$, a DEX investment level $I^{\star}$, a DEX USD inventory level $I^{\star}_{USD}$, a DEX ETH inventory level $I^{\star}_{ETH}$, a DEX pricing function $P^{\star}_{DEX}(\delta) := \Xi(I^{\star}_{USD}, I^{\star}_{ETH}, \delta)$, and a set of traders that trade at the DEX $D^{\star}$ such that each trader optimally selects the exchange at which she trades and each investor invests in the DEX iff it is optimal. More explicitly, an equilibrium is defined by the requirement that Equations (1) - (8) must all hold

[3] We follow prior literature (e.g., Capponi and Jia 2021 and Lehar and Parlour 2021) and assume that DEX fee revenues are held in a separate account than DEX inventory, being distributed directly from that account to investors.

simultaneously with the equilibrium solutions replacing the associated endogenous object in each equation.

Our main focus is on examining the implications of the DEX fee level, f_{DEX}, upon equilibrium objects. As such, we explicitly state the dependence of all equilibrium objects on the DEX fee level. Moreover, we omit discussion regarding the trivial case of $f_{DEX} > f_{CEX}$ and restrict ourselves to $f_{DEX} \in [0, f_{CEX}]$ given that $f_{DEX} > f_{CEX}$ trivially implies that DEX trading costs exceed CEX trading costs for all traders so that no trading occurs at the DEX in equilibrium unless $f_{DEX} \leqslant f_{CEX}$.

Turning to our equilibrium solution, we solve for a symmetric equilibrium in that we require that all liquidity traders of the same size must trade at the same exchange. Explicitly, we require that the set of liquidity traders trading at the DEX, $D^\star$, is of the following form:

$$D^\star = \{j : \delta_j \in \Delta^\star\} \tag{9}$$

where $\Delta^\star \subseteq [-1, 1]$ denotes the range of trade sizes such that a liquidity trader trades at the DEX with $[-1, 1]$ being the support for the distribution that generates trade sizes.

As an intermediate step to solving for an equilibrium, we begin with the following result that derives the optimal behaviour of the liquidity traders while taking as given the investment level of the DEX, $I^\star(f_{DEX})$:

Proposition 2.1. *Optimal Trading Strategy*

Denote by $I^\star(f_{DEX})$ the equilibrium DEX investment level. The optimal strategy for Liquidity Trader j is:

$$i^\star(j) = \begin{cases} DEX & \text{if } \delta_j \in \Delta^\star(f_{DEX}) \\ CEX & \text{Otherwise} \end{cases} \tag{10}$$

where $\Delta^\star(f_{DEX}) := [\delta^\star_-(f_{DEX}), \delta^\star_+(f_{DEX})]$ with the bounds $\delta^\star_-(f_{DEX}) < 0$ and $\delta^\star_+(f_{DEX}) > 0$ given explicitly as:

$$\delta^\star_-(f_{DEX}) = -\frac{(f_{CEX} - f_{DEX})}{1 - (f_{CEX} - f_{DEX})} \cdot \frac{I^\star(f_{DEX})}{2V} \tag{11}$$

and

$$\delta_+^\star(f_{DEX}) = \frac{f_{CEX} - f_{DEX}}{1 + (f_{CEX} - f_{DEX})} \cdot \frac{I^\star(f_{DEX})}{2V} \tag{12}$$

The endogenous quantities given by Equations (11) and (12), $\delta_+^\star(f_{DEX})$ and $\delta_-^\star(f_{DEX})$, determine the equilibrium buy and sell cut-off sizes respectively in that traders with an ETH buy order (i.e., $\delta > 0$) trade at the DEX if and only if $\delta \leqslant \delta_+^\star(f_{DEX})$, whereas traders with an ETH sell order (i.e., $\delta < 0$) trade at the DEX if and only if $\delta \geqslant \delta_-^\star(f_{DEX})$. This derived structure, characterized by cut-offs, implies that traders with larger trade sizes (in absolute magnitude) prefer trading at the CEX relative to the DEX, which is a necessary feature of any symmetric equilibrium. In particular, the DEX ETH price is increasing in trade size (i.e., $\frac{dP_{DEX}(\delta)}{d\delta} > 0$), whereas the CEX ETH price is always equal to fair value (i.e., $P_{CEX}(\delta) = V$ and thus $\frac{dP_{CEX}(\delta)}{d\delta} = 0$) so that the DEX entails an increasing average cost of trading (i.e., $\frac{\Psi_{DEX}(\delta)}{|\delta|}$ increases in $|\delta|$) while the CEX entails a constant average cost of trading (i.e., $\frac{\Psi_{CEX}(\delta)}{|\delta|}$ is constant in $|\delta|$). Hence, the DEX necessarily will only be optimal for smaller trade sizes (i.e., $\Delta^\star(f_{DEX})$ must be of the form $[\delta_-^\star(f_{DEX}), \delta_+^\star(f_{DEX})]$).

Our next result builds upon Proposition 2.1, deriving a unique equilibrium:[4]

Proposition 2.2. *Unique Equilibrium*

There exists a unique non-trivial symmetric equilibrium which is given as follows:

- *Equilibrium Investment Return at the DEX*

  The equilibrium expected return from investing in the DEX is:

$$r_{DEX}^\star(f_{DEX}) = \left(\frac{f_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2} \right)^{\frac{\theta}{\theta - 1}} \tag{13}$$

- *Equilibrium Investment Level at the DEX*

[4] As a technical aside, Proposition 2.2 establishes a unique non-trivial equilibrium where a non-trivial equilibrium is defined as an equilibrium that features non-zero DEX trading volume. Note that there always exists a trivial equilibrium with zero DEX trading volume. In particular, if the DEX were to possess no investment, then trading costs would be infinite, no trading would occur at the DEX, and therefore investment returns would be zero, supporting the optimality of zero investment and ensuring that such a trivial equilibrium always exists. We omit discussion regarding this trivial equilibrium because its properties are straight-forward and well-known in the more general context of settings with positive network effects.

The equilibrium DEX investment level is:

$$I^\star(f_{DEX}) = \left(\frac{f_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2}\right)^{\frac{1}{\theta - 1}} \tag{14}$$

- *Equilibrium Inventory Levels at DEX*

  Equilibrium inventory of ETH and USD are functions of DEX investment as follows:

$$I^\star_{USD}(I) = \frac{I}{2}, \qquad I^\star_{ETH}(I) = \frac{I}{2V} \tag{15}$$

so that applying the equilibrium investment level from Equation (14) to Equation (15) yields the explicit equilibrium inventory solutions:

$$I^\star_{ETH}(f_{DEX}) = \frac{1}{2V}\left(\frac{f_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2}\right)^{\frac{1}{\theta - 1}} \tag{16}$$

and

$$I^\star_{USD}(f_{DEX}) = \frac{1}{2}\left(\frac{f_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2}\right)^{\frac{1}{\theta - 1}} \tag{17}$$

- *Equilibrium Pricing Function at DEX*

  The equilibrium DEX pricing function depends upon DEX investment as follows:

$$P_{DEX}(I, \delta) = \Xi(I^\star_{USD}(I), I^\star_{ETH}(I), \delta) \tag{18}$$

where $\Xi(\cdot,\cdot,\cdot)$, $I^\star_{USD}(I)$ and $I^\star_{ETH}(I)$ are given by Equations (3) and (15).

In turn, applying the equilibrium investment level from Equation (14) to Equation (18) yields the equilibrium DEX pricing function:

$$P^\star_{DEX}(f_{DEX}, \delta) := P_{DEX}(I^\star(f_{DEX}), \delta) = \Xi(\frac{1}{2}I^\star(f_{DEX}), \frac{1}{2V}I^\star(f_{DEX}), \delta) \tag{19}$$

Since our primary object of interest is the equilibrium trading volume, we provide the following corollary which derives the equilibrium trading volume, $T^\star(f_{DEX})$:

Corollary 2.3. *Equilibrium Trading Volume*

The equilibrium expected trading volume (in USD), $T^\star(f_{DEX})$, is given as follows:

$$T^\star(f_{DEX}) = \frac{V}{2} \cdot (\delta_+^\star(f_{DEX})^2 + \delta_-^\star(f_{DEX})^2) \tag{20}$$

with $\delta_-^\star(f_{DEX})$ and $\delta_+^\star(f_{DEX})$ being given in Equations (11) and (12) respectively.

3 Results

We begin with our main result, Proposition 3.1, which establishes that increases in fees charged to traders at the DEX can increase the equilibrium DEX trading volume:

Proposition 3.1. *DEX Trading Volume Can Increase in DEX fees*

The equilibrium expected trading volume, $T^\star(f_{DEX})$, first increases and then decreases in the DEX fee level, f_{DEX}. More formally, there exists $\tilde{f} \in (0, f_{CEX})$ such that $\frac{dT^\star}{df_{DEX}} > 0$ for $f_{DEX} \in (0, \tilde{f})$, whereas $\frac{dT^\star}{df_{DEX}} < 0$ for $f_{DEX} \in (\tilde{f}, f_{CEX})$.

This result is notable because it highlights a significant economic distinction between a DEX and a CEX. In more detail, an increase in trading fees at a CEX would unambiguously reduce trading volume at that CEX, and Proposition 3.1 thereby distinguishes a DEX from a CEX by demonstrating that the referenced unambiguous negative relationship between trading fees and trading volume does not apply for a DEX. Explicitly, Proposition 3.1 establishes that there exists a non-zero fee level, $\tilde{f}$, such that increases in the DEX fee level up to $\tilde{f}$ will always lead to increases in the DEX trading volume.

Proposition 3.1 arises because an increase in the DEX fee level can decrease the overall cost of trading at the DEX. In turn, since a trader optimally trades at the exchange that charges her the lowest trading cost (see Equation 5), an increase in the DEX fee level can generate increases in trading volume as per Proposition 3.1 specifically because such DEX fee increases reduce DEX trading costs. We formalize this point with our next result:

Proposition 3.2. *DEX Trading Costs Can Decrease in DEX Fees*

Let $\Psi_{DEX}^\star(f_{DEX}, \delta)$ denote the equilibrium DEX trading cost for a trader with trade size δ, given *explicitly as:*

$$\Psi^{\star}_{DEX}(f_{DEX}, \delta) = P^{\star}_{DEX}(f_{DEX}, \delta) \times \delta + f_{DEX} \times |\delta| \times V \tag{21}$$

where $P^{\star}_{DEX}(f_{DEX}, \delta)$ *is given by Equation* (19).

Then, the following results hold:

1.) There exists $\hat{f}_{+} \in (0, f_{CEX})$ *such that cost of trading with the DEX for the marginal buy trader (i.e., the trader with size* $\delta^{\star}_{+}(f_{DEX}) > 0$*) is decreasing in* f_{DEX} *when* $f_{DEX} \in (0, \hat{f}_{+})$ *and increasing in* f_{DEX} *when* $f_{DEX} \in (\hat{f}_{+}, f_{CEX})$*. In particular,* $\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta^{\star}_{+}(f_{DEX}))} < 0$ *when* $f_{DEX} \in (0, \hat{f}_{+})$ *and* $\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta^{\star}_{+}(f_{DEX}))} > 0$ *when* $f_{DEX} \in (\hat{f}_{+}, f_{CEX})$.

2.) There exists $\hat{f}_{-} \in (0, f_{CEX})$ *such that cost of trading with the DEX for the marginal sell trader (i.e., the trader with size* $\delta^{\star}_{-}(f_{DEX}) < 0$*) is decreasing in* f_{DEX} *when* $f_{DEX} \in (0, \hat{f}_{-})$ *and increasing in* f_{DEX} *when* $f_{DEX} \in (\hat{f}_{-}, f_{CEX})$*. In particular,* $\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta^{\star}_{-}(f_{DEX}))} < 0$ *when* $f_{DEX} \in (0, \hat{f}_{-})$ *and* $\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta^{\star}_{-}(f_{DEX}))} > 0$ *when* $f_{DEX} \in (\hat{f}_{-}, f_{CEX})$.

Proposition 3.2 establishes that an increase in the DEX fee level can reduce the overall DEX trading cost for a trader who would have been indifferent between trading at the DEX and trading at the CEX in the absence of such a DEX fee level increase. This result focuses upon traders who would have been indifferent between the DEX and the CEX in the absence of the DEX fee level change because changes in the trading costs of such marginal traders directly imply changes in our primary object of interest, the DEX trading volume. More specifically, if the DEX trading cost falls for a trader who would have been indifferent between the DEX and the CEX in the absence of the DEX fee level increase, then that trader strictly prefers to trade at the DEX as opposed to the CEX after the DEX fee level increases. When this is the case, the trade size of the marginal trader (in absolute magnitude) must increase in response to an increase in DEX fees. In turn, an increase in the DEX fee level can generate an increase in DEX trading volume as per Proposition 3.1 precisely because the increase in the DEX fee level decreases DEX trading costs as per Proposition 3.2.

To understand Proposition 3.2, we reiterate that fees are not the only cost associated with trading at a DEX. In particular, Equation (1) highlights that the overall trading cost depends not only on the fee f_{DEX} but also on the price at which the cryptoasset is being traded $P^{\star}_{DEX}(f_{DEX}, \delta)$. Therefore, if an increase in the DEX fee level leads to a lower price impact (i.e., if $\frac{\partial P^{\star}_{DEX}}{\partial \delta}$ decreases in f_{DEX}), then an increase in the DEX fee level will reduce the overall DEX trading cost provided that the cost of paying a higher fee can be offset by the decrease in the trading cost due to trading at a price that is more favorable (i.e., a lower price impact). Our next result formally establishes such a channel whereby an increase in the DEX fee level reduces the DEX price impact so long as DEX fees are initially not too large:

Proposition 3.3. *DEX Price Impacts Can Decrease in DEX fees*

The equilibrium price impact at the DEX first decreases and then increases in the DEX fee level. More formally, letting

$$\lambda^{\star}(f_{DEX}, \delta) = \frac{\partial P^{\star}_{DEX}(f_{DEX}, \delta)}{\partial \delta} \tag{22}$$

denote the equilibrium price impact at the DEX. Then, there exists $\tilde{f} \in (0, f_{CEX})$ *such that* $\frac{\partial \lambda^{\star}}{\partial f_{DEX}} < 0$ *for* $f_{DEX} \in (0, \tilde{f})$ *and* $\frac{\partial \lambda^{\star}}{\partial f_{DEX}} > 0$ *for* $f_{DEX} \in (\tilde{f}, f_{CEX})$*. This* $\tilde{f}$ *applies uniformly for all feasible trade sizes (i.e., for* $\delta < I^{\star}_{ETH}$*).*

Proposition 3.3 defines equilibrium DEX price impact as the sensitivity of the DEX price to trade size (see Equation 22), and then establishes that the DEX price impact is decreasing in the DEX fee level whenever the initial DEX fee level is sufficiently small (i.e., $\frac{\partial \lambda^{\star}}{\partial f_{DEX}} < 0$ for $f \in (0, \tilde{f})$).

The relationship between DEX fee levels and price impact is important because it affects the overall DEX trading cost which in turn affects our main object of interest, DEX trading volume. In particular, DEX trading prices mechanically move in the direction of a trade (see Equation 3) so that a larger price impact (i.e., a larger $\lambda^{\star}$) entails that a given buy order (i.e., $\delta > 0$) would involve a higher price and also entails that a given sell order (i.e., $\delta < 0$) would involve a lower price. As the trading cost is increasing in the price for a buy order but decreasing in the price for a sell order (see Equation 1), a larger price impact entails a higher cost for all traders. Thus, the result of Proposition 3.3, that price impacts decline for DEX fee levels up to a point (i.e., $\frac{\partial \lambda^{\star}}{\partial f_{DEX}} < 0$ for $f \in (0, \tilde{f})$), clarifies that increases in the DEX fee level can reduce the execution price component of the DEX trading cost. This is precisely the mechanism whereby increases in the DEX fee level can reduce overall DEX trading costs (Proposition 3.2) and also increase DEX trading volume (Proposition 3.1).

The relationship that Proposition 3.3 establishes between the DEX fee level and the DEX price impact arises due to two intermediate relationships. First, the mechanical pricing rule of a DEX (i.e., Equation 3) implies that an increase in total DEX investment always reduces DEX price impacts. Second, all DEX fee revenues are paid to investors which creates the incentive for investors to provide DEX investment (see Equation 7) so that increases in the DEX fee level can lead to increases in overall DEX investment. Then, putting the two aforementioned relationships together, an increase in the DEX fee level can generate increases in total DEX investment which, in turn, reduces DEX price impacts (Proposition 3.3) and therefore promotes higher DEX trading volume (Proposition 3.1). We proceed by formalizing the referenced intermediate relationships with Proposition 3.4 demonstrating the first relationship that increases in DEX investment decrease DEX price impacts, and Propositions 3.5 - 3.6 establishing the second relationship that increases in the DEX fee level can increase DEX investment.

Proposition 3.4. *DEX Price Impacts Always Decrease in DEX Investment*

The price impact is monotonically decreasing in the DEX investment level for all feasible trade sizes. More explicitly, $\frac{\partial \lambda(I,\delta)}{\partial I} < 0$ *for all investment levels,* I*, and for all feasible trade sizes,* $\delta < I^{\star}_{ETH}(I) = \frac{I}{2V}$*, where* $\lambda(I,\delta)$ *denotes the price impact given an arbitrary investment level,* $I > 0$*:*

$$\lambda(I,\delta) = \frac{\partial P_{DEX}(I,\delta)}{\partial \delta}$$

with $P_{DEX}(I,\delta)$ *being defined in Equation (18).*

Proposition 3.4 establishes that an increase in the DEX investment level unambiguously reduces the DEX price impact (i.e., $\frac{\partial \lambda}{\partial I} < 0$). This result arises due to the mechanical pricing function of the DEX (see Equation 3). To clarify this point, note that $P_{DEX}(I,\delta)$, which is defined in Equation (18), can be derived explicitly from Equations (3) and (15) as follows:

$$P_{DEX}(I,\delta) = \Xi(I^{\star}_{USD}(I), I^{\star}_{ETH}(I), \delta) = \frac{I \cdot V}{I - 2 \cdot V \cdot \delta} \tag{23}$$

In turn, the price impact, $\lambda(I,\delta)$, as a function of DEX investment, I, and trade size, δ, is given as follows:

$$\lambda(I,\delta) = \frac{\partial P_{DEX}(I,\delta)}{\partial \delta} = \frac{2 \cdot I \cdot V^2}{(I - 2 \cdot V \cdot \delta)^2} \tag{24}$$

so that direct verification reveals that the DEX price impact monotonically decreases in investment (i.e., $\frac{\partial \lambda}{\partial I} < 0$) whenever the trade size is feasible (i.e., when $\delta < I^{\star}_{ETH}(I) = \frac{I}{2V}$) as per Proposition 3.4. We ignore $\delta > I^{\star}_{ETH}(I) = \frac{I}{2V}$ and label such trade sizes as infeasible because in such a case there is insufficient inventory for the trade size and the price is consequently infinite (see Equation 3), which ensures that the DEX would not allow a trade of such size.

Having established that increases in DEX investment decrease DEX price impacts (Proposition 3.4), we turn to demonstrating that increases in the DEX fee level can increase DEX investment:

Proposition 3.5. *DEX Investment Can Increase in DEX Fee Levels*
The equilibrium DEX investment, $I^{\star}(f_{DEX})$, first increases and then decreases in the DEX fee level, f_{DEX}. More formally, there exists $\tilde{f} \in (0, f_{CEX})$ such that $\frac{dI^{\star}}{df_{DEX}} > 0$ for $f_{DEX} \in (0, \tilde{f})$, whereas $\frac{dI^{\star}}{df_{DEX}} < 0$ for $f_{DEX} \in (\tilde{f}, f_{CEX})$ with $I^{\star}(f_{DEX})$ being given in Equation (14).

Proposition 3.5 establishes that increases in the DEX fee level can increase DEX investment up to some fee level $\tilde{f}$ (i.e., $\frac{dI^{\star}}{df_{DEX}} > 0$ for $f_{DEX} \in (0, \tilde{f})$). This result arises because DEXs acquire investment by offering investors a pro-rata share of all trading fees from the DEX in exchange for those investments (see Equation 7); in particular, for $f < \tilde{f}$, an increase in the DEX fee level increases DEX investment by increasing the DEX investment return through an increase in the overall trading fee revenue generated by the DEX. We formalize the point that such increases in the DEX fee level generate an increase in the DEX investment return with our final result:

Proposition 3.6. *DEX Investment Returns Can Increase in DEX Fee Levels*
The equilibrium DEX investment return, $r^{\star}_{DEX}(f_{DEX})$, first increases and then decreases in the DEX fee level. More formally, there exists $\tilde{f} \in (0, f_{CEX})$ such that $\frac{dr^{\star}_{DEX}}{df_{DEX}} > 0$ for $f_{DEX} \in (0, \tilde{f})$, whereas $\frac{dr^{\star}_{DEX}}{df_{DEX}} < 0$ for $f_{DEX} \in (\tilde{f}, f_{CEX})$.

Collectively, our results establish an important distinguishing feature between a DEX and a CEX; more precisely, we demonstrate that an increase in fees at a DEX can increase trading volume at the DEX, whereas an increase in fees at a CEX would necessarily reduce trading volume at the CEX. Our main result, Proposition 3.1, establishes this finding, whereas our remaining results clarify the associated economic channel. In more detail, an increase in the DEX fee level can increase DEX investment returns (Proposition 3.6) and thereby DEX investment (Proposition 3.5), which generates a reduction in the DEX price impact (Propositions 3.3 and 3.4) and thereby a reduction in DEX trading costs (Proposition 3.2). In turn, the reduction in DEX trading costs drives trading activity from the CEX to the DEX, leading to an increase in equilibrium DEX trading volume as per Proposition 3.1.

4 Conclusion

We provide an economic model of a DEX. Our model is specifically aimed at clarifying the implications of varying DEX fee levels upon equilibrium quantities such as DEX trading volume and DEX trading costs. Of particular note, we demonstrate that increases in DEX fee levels can reduce DEX trading costs and thereby increase DEX trading volume. The referenced result is especially notable because it does not arise for a CEX and thereby highlights a novel economic channel which distinguishes DEXs from traditional exchanges.

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Appendices

A Supplementary Results

Lemma A.1.

Let $\delta : (0, f_{CEX}) \mapsto \mathbb{R}$ denote any non-zero continuously differentiable function that satisfies:

$$\Psi^{\star}_{DEX}(f_{DEX}, \delta(f_{DEX})) = \beta \times \delta(f_{DEX}) \tag{A.1}$$

for all $f_{DEX} \in (0, f_{CEX})$ and for any $\beta \in \mathbb{R}$ where $\Psi^{\star}_{DEX}$ refers to $\Psi^{\star}_{DEX}(f_{DEX}, \delta)$ which is given by Equation (21). Then, the following result holds:

$$\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}} = -\delta(f_{DEX}) \times \frac{\partial P^{\star}_{DEX}(f_{DEX}, \delta)}{\partial \delta} \times \frac{d\delta}{df_{DEX}}$$

for all $f_{DEX} \in (0, f_{CEX})$ where $\Psi^{\star}_{DEX}$ and $P^{\star}_{DEX}$ are each evaluated at $(f_{DEX}, \delta(f_{DEX}))$.

Proof. We begin by taking the total derivative in Equation (A.1) with respect to f_{DEX} which yields:

$$\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}} + \frac{\partial \Psi^{\star}_{DEX}}{\partial \delta} \times \frac{d\delta}{df_{DEX}} = \beta \times \frac{d\delta}{df_{DEX}} \tag{A.2}$$

and further implies:

$$\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}} = \left(\beta - \frac{\partial \Psi^{\star}_{DEX}}{\partial \delta}\right) \times \frac{d\delta}{df_{DEX}} \tag{A.3}$$

By explicit calculation, Equation (21) yields:

$$\frac{\partial \Psi^{\star}_{DEX}}{\partial \delta} = \Big(P^{\star}_{DEX}(f_{DEX}, \delta(f_{DEX})) + f_{DEX} \times V\Big) + \delta(f_{DEX}) \times \frac{\partial P^{\star}_{DEX}}{\partial \delta} \tag{A.4}$$

whereas Equation (A.1) is equivalent to:

$$P^{\star}_{DEX}(f_{DEX}, \delta(f_{DEX})) + f_{DEX} \times V = \beta \tag{A.5}$$

so that applying Equation (A.5) to Equation (A.4) and then applying the result to Equation (A.3) yields:

$$\frac{\partial \Psi^{\star}_{DEX}}{\partial f_{DEX}} = -\delta(f_{DEX}) \times \frac{\partial P^{\star}_{DEX}}{\partial \delta} \times \frac{d\delta}{df_{DEX}} \tag{A.6}$$

thereby completing the proof. □

B Proofs

B.1 Proof of Proposition 2.1

A liquidity trader with trade size $\delta \in \mathbb{R}$ trades with the DEX if and only if the cost of doing so is less than the cost of trading with the CEX. This is the case if and only if

$$P^{\star}_{DEX}(\delta) \cdot \delta + f_{DEX} \cdot |\delta| \cdot V \leqslant V \cdot \delta + f_{CEX} \cdot |\delta| \cdot V$$

Therefore, $\delta^{\star}_{-}(f_{DEX}) < 0$ is the trade size of a liquidity trader that wishes to sell ETH and is indifferent between trading at the DEX and CEX, given by:

$$\frac{I_{USD}}{I_{ETH} - \delta^{\star}_{-}(f_{DEX})} \cdot \delta^{\star}_{-}(f_{DEX}) - f_{DEX} \cdot \delta^{\star}_{-}(f_{DEX}) \cdot V = V \cdot \delta^{\star}_{-}(f_{DEX}) - f_{CEX} \cdot \delta^{\star}_{-}(f_{DEX}) \cdot V$$

which after solving for $\delta^{\star}_{-}(f_{DEX})$ using the fact that $I^{\star}_{USD} = V \cdot I^{\star}_{ETH}$ and $I^{\star}_{ETH} = \frac{I^{\star}(f_{DEX})}{2 \cdot V}$ yields our expression for $\delta^{\star}_{-}(f_{DEX})$.

Similarly, $\delta_+^\star(f_{DEX}) > 0$ is the trade size of a liquidity trader that wishes to buy ETH and is indifferent between trading at the DEX and CEX, given by:

$$\frac{I_{USD}}{I_{ETH} - \delta_+^\star(f_{DEX})} \cdot \delta_+^\star(f_{DEX}) + f_{DEX} \cdot \delta_+^\star(f_{DEX}) \cdot V = V \cdot \delta_+^\star(f_{DEX}) + f_{CEX} \cdot \delta_+^\star(f_{DEX}) \cdot V$$

which again after substituting and rearranging and using the fact that $I_{USD}^\star = V \cdot I_{ETH}^\star$ and $I_{ETH}^\star = \frac{I^\star(f_{DEX})}{2 \cdot V}$ yields our expression for $\delta_+^\star(f_{DEX})$.

B.2 Proof of Proposition 2.2

Proof. To solve for the equilibrium return $r_{DEX}^\star(f_{DEX})$ we start by rearranging (7) to obtain

$$r_{DEX} = \frac{2 \cdot V \cdot f_{DEX}}{G(r_{DEX})} \cdot \mathbb{E}[\sum_{j \in \mathcal{D}} |\delta_j|]$$

Then, noting that $\mathcal{D} = \{j : \delta_j \in [\bar{\delta}_-, \bar{\delta}_+]\}$ and $N \sim Poisson(1)$ implies that the expected number of trades are $Pr(\delta_j \in [\delta_-^\star, \delta_+^\star])$ and therefore

$$\mathbb{E}[\sum_{j \in \mathcal{D}} |\delta_j|] = Pr(\delta_j \in [\delta_-^\star, \delta_+^\star]) \cdot \mathbb{E}[|\delta_j| \mid \delta_j \in [\delta_-^\star, \delta_+^\star]] = \frac{(\delta_+^\star)^2 + (\delta_-^\star)^2}{4}$$

Therefore, using the fact that

$$(\delta_+^\star)^2 + (\delta_-^\star)^2 = (\frac{(f_{CEX} - f_{DEX})^2}{(1 + (f_{CEX} - f_{DEX}))^2} + \frac{(f_{CEX} - f_{DEX})^2}{(1 - (f_{CEX} - f_{DEX}))^2}) \cdot (\frac{I^\star(f_{DEX})}{2V})^2$$

then implies

$$r_{DEX} = \frac{2 \cdot V \cdot f_{DEX}}{G(r_{DEX})} \cdot \frac{1}{4}(\frac{(f_{CEX} - f_{DEX})^2}{(1 + (f_{CEX} - f_{DEX}))^2} + \frac{(f_{CEX} - f_{DEX})^2}{(1 - (f_{CEX} - f_{DEX}))^2}) \cdot (\frac{I^\star(f_{DEX})}{2V})^2$$

and using $I^\star(f_{DEX}) = G(r_{DEX}^\star) = (r_{DEX}^\star)^{\frac{1}{\theta}}$, then after rearranging we obtain

$$r_{DEX}^\star = \left(\frac{f_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2}\right)^{\frac{\theta}{\theta - 1}}$$

Finally, substituting $r^\star_{DEX}$ into $I^\star(f_{DEX}) = (r^\star_{DEX})^{\frac{1}{\theta}}$ yields

$$I^\star(f_{DEX}) = \left(\frac{f_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2}\right)^{\frac{1}{\theta-1}}$$

Finally, note that (4) implies that inventory must be deposited in the ratio of 1 USD per $\frac{1}{V}$ ETH and given that each investor is born with a unit of USD capital then they must split that by providing $\frac{1}{2}$ USD and $\frac{1}{V}$ ETH to the DEX. Therefore, $I^\star_{ETH} = \frac{1}{2V} I^\star(f_{DEX})$ and $I^\star_{USD} \frac{1}{2} I^\star(f_{DEX})$.

□

B.3 Proof of Proposition 3.1

Proof. First note that

$$T^\star(f_{DEX}) = \frac{G(r^\star_{DEX})}{f_{DEX}} \cdot r^\star_{DEX} = \left(\frac{f^\alpha_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{8V((1 - (f_{CEX} - f_{DEX})^2)^2}\right)^{\frac{\theta+1}{\theta-1}}$$

where $\alpha = \frac{2}{\theta+1} < 1$. Next, denote by $g(f_{DEX})$ the following function

$$g(f_{DEX}) := \frac{f^\alpha_{DEX}(f_{CEX} - f_{DEX})^2(1 + (f_{CEX} - f_{DEX})^2)}{((1 - (f_{CEX} - f_{DEX})^2)^2}$$

Note that $g(f_{DEX}) > 0$ for $f_{DEX} \in (0, f_{CEX})$ and $g(0) = g(f_{CEX}) = 0$. Therefore, proving the result only requires showing that $g(f_{DEX})$ has a unique local maximum on $[0, f_{CEX}]$. In order to do so, we will show that $log(g(f_{DEX}))$ has a unique local maximum on $(0, f_{CEX})$ which implies that $g(f_{DEX})$ must have a unique local maximum on $(0, f_{CEX})$. In particular, we will do this by showing that $log(g(f_{DEX}))$ is strictly concave (i.e. $\frac{\partial^2}{\partial f^2_{DEX}} log(g(f_{DEX})) < 0$). First, note that

$$log(g(f_{DEX})) = \alpha log(f_{DEX}) + 2 log(f_{CEX} - f_{DEX}) + log(1 + (f_{CEX} - f_{DEX})^2) - 2 log(1 - (f_{CEX} - f_{DEX})^2)$$

so that

$$\frac{\partial}{\partial f_{DEX}} log(g(f_{DEX})) = \frac{\alpha}{f_{DEX}} - \frac{2}{f_{CEX} - f_{DEX}} - \frac{2(f_{CEX} - f_{DEX})}{1 + (f_{CEX} - f_{DEX})^2} - \frac{4(f_{CEX} - f_{DEX})}{1 - (f_{CEX} - f_{DEX})^2}$$

and therefore

$$\frac{\partial^2}{\partial f_{DEX}^2} log(g(f_{DEX})) = -\frac{\alpha}{f_{DEX}^2} - \frac{2}{(f_{CEX}-f_{DEX})^2} + \frac{2(1-(f_{CEX}-f_{DEX})^2)}{(1+(f_{CEX}-f_{DEX})^2)^2} + \frac{4(1+(f_{CEX}-f_{DEX})^2)}{(1-(f_{CEX}-f_{DEX})^2)^2}$$

Finally, note that $-\frac{\alpha}{f_{DEX}^2} \leqslant 0$ and $(f_{CEX}-f_{DEX})^2 \leqslant \overline{f}^2$, thereby implying:

$$\frac{\partial^2}{\partial f_{DEX}^2} log(g(f_{DEX})) \leqslant \sup_{z:z\in[0,\overline{f}^2]} \Big(-\frac{2}{z} + \frac{2(1-z)}{(1+z)^2} + \frac{4(1+z)}{(1-z)^2} \Big) < 0 \tag{A.7}$$

where the last inequality follows from direct verification by using $\overline{f} = 25\%$.

□

B.4 Proof of Proposition 3.2

Proof. In order to prove this result, we will prove that both $\delta_+^\star(f_{DEX})$ and $-\delta_-^\star(f_{DEX})$ each have a unique local maximum. In order to do so, we will prove that $log(\delta_+^\star(f_{DEX}))$ and $log(-\delta_-^\star(f_{DEX}))$ are concave and therefore each have a unique local maximum, implying that $\delta_+^\star(f_{DEX})$ and $-\delta_-^\star(f_{DEX})$ each have a unique local maximum. In order to do so, first note that

$$log(\delta_+^\star(f_{DEX})) = log(\frac{f_{CEX}-f_{DEX}}{1+f_{CEX}-f_{DEX}}) + log(I^\star(f_{DEX})) - log(2V)$$

and

$$log(-\delta_-^\star(f_{DEX})) = log(\frac{f_{CEX}-f_{DEX}}{1-(f_{CEX}-f_{DEX})}) + log(I^\star(f_{DEX})) - log(2V)$$

Next, we note that

$$\frac{\partial}{\partial f_{DEX}} log(\frac{f_{CEX}-f_{DEX}}{1+f_{CEX}-f_{DEX}}) = \frac{-1}{f_{CEX}-f_{DEX}} + \frac{1}{1+f_{CEX}-f_{DEX}}$$

$$\frac{\partial^2}{\partial f_{DEX}^2} log(\frac{f_{CEX}-f_{DEX}}{1+f_{CEX}-f_{DEX}}) = \frac{-1}{(f_{CEX}-f_{DEX})^2} + \frac{1}{(1+f_{CEX}-f_{DEX})^2} < 0$$

$$\frac{\partial}{\partial f_{DEX}} log(\frac{f_{CEX}-f_{DEX}}{1-(f_{CEX}-f_{DEX})}) = \frac{-1}{f_{CEX}-f_{DEX}} - \frac{1}{1-(f_{CEX}-f_{DEX})}$$

$$\frac{\partial^2}{\partial f_{DEX}^2} log(\frac{f_{CEX}-f_{DEX}}{1-(f_{CEX}-f_{DEX})}) = \frac{-1}{(f_{CEX}-f_{DEX})^2} + \frac{1}{(1-(f_{CEX}-f_{DEX}))^2} < 0$$

where the last inequality holds whenever $f_{CEX} - f_{DEX} < .5$ which is guaranteed to hold given that we have assumed that $f_{CEX} < \bar{f} \leqslant .25$.

Next, using (14) we can see that

$$\frac{\partial}{\partial f_{DEX}} log(I^\star(f_{DEX})) = (\frac{1}{\theta-1})(\frac{1}{f_{DEX}} - \frac{2}{f_{CEX}-f_{DEX}} - \frac{2(f_{CEX}-f_{DEX})}{1+(f_{CEX}-f_{DEX})^2} - \frac{4(f_{CEX}-f_{DEX})}{1-(f_{CEX}-f_{DEX})^2})$$

and therefore

$$\frac{\partial^2}{\partial f_{DEX}^2} log(I^\star(f_{DEX})) = (\frac{1}{\theta-1})(-\frac{1}{f_{DEX}^2} - \frac{2}{(f_{CEX}-f_{DEX})^2} + \frac{2(1-(f_{CEX}-f_{DEX})^2)}{(1+(f_{CEX}-f_{DEX})^2)^2} + \frac{4(1+(f_{CEX}-f_{DEX})^2)}{(1-(f_{CEX}-f_{DEX})^2)^2})$$

Finally, ote that $-\frac{1}{f_{DEX}^2} \leqslant 0$ and $(f_{CEX} - f_{DEX})^2 \leqslant \overline{f}^2$, thereby implying:

$$\frac{\partial^2}{\partial f_{DEX}^2} log(I^\star(f_{DEX})) \leqslant \frac{1}{\theta-1} \sup_{z:z\in[0,\overline{f}^2]} \Big(-\frac{2}{z} + \frac{2(1-z)}{(1+z)^2} + \frac{4(1+z)}{(1-z)^2}\Big) < 0$$

where the last inequality follows from direct verification by using $\overline{f} = 25\%$.

What we have shown is that there exists $\tilde{f}_+$ and $\tilde{f}_-$ such that $\delta_+^\star(f_{DEX})$ is increasing in f_{DEX} for $f_{DEX} \in [0, \tilde{f}_+)$ and decreasing in f_{DEX} for $f_{DEX} \in (\tilde{f}_+, f_{CEX}]$ while $\delta_-^\star(f_{CEX})$ is decreasing for $f_{DEX} \in [0, \tilde{f}_-)$ and increasing for $f_{DEX} \in (\tilde{f}_-, f_{CEX}]$.

In order to conclude the proof, we apply Lemma A.1 to $\delta_+^\star(f)$ for $\beta = V \times (1 + f_{CEX})$ and apply Lemma A.1 to $\delta_-^\star(f)$ for $\beta = V \times (1 - f_{CEX})$ which yields:

$$\frac{\partial \Psi_{DEX}^\star}{\partial f_{DEX}} = -\delta_+^\star(f_{DEX}) \frac{\partial P_{DEX}^\star(f_{DEX},\delta)}{\partial \delta} \frac{d\delta_+^\star}{df_{DEX}}, \qquad \frac{\partial \Psi_{DEX}^\star}{\partial f_{DEX}} = -\delta_-^\star(f_{DEX}) \frac{\partial P_{DEX}^\star(f_{DEX},\delta)}{\partial \delta} \frac{d\delta_-^\star}{df_{DEX}}$$

Finally, we note that $\frac{\partial P_{DEX}^\star(f_{DEX},\delta)}{\partial \delta} > 0$ coupled with $\delta_+^\star(f_{DEX}) > 0$ when combined with the aforementioned result on the sign of $\frac{d\delta_+^\star}{df_{DEX}}$ implies that $\frac{\partial \Psi_{DEX}^\star}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta_+^\star(f_{DEX}))} < 0$ for all $f_{DEX} \in (0, \hat{f}_+)$ and $\frac{\partial \Psi_{DEX}^\star}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta_+^\star(f_{DEX}))} > 0$ for all $f_{DEX} \in (\hat{f}_+, f_{CEX})$. Similarly, $\frac{\partial P_{DEX}^\star(f_{DEX},\delta)}{\partial \delta} > 0$ coupled with $\delta_-^\star(f_{DEX}) < 0$ when combined with the aforementioned result on the sign of $\frac{d\delta_+^\star}{df_{DEX}}$ implies that $\frac{\partial \Psi_{DEX}^\star}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta_-^\star(f_{DEX}))} < 0$ for all $f_{DEX} \in (0, \hat{f}_-)$ and $\frac{\partial \Psi_{DEX}^\star}{\partial f_{DEX}}|_{(f_{DEX},\delta)=(f_{DEX},\delta_-^\star(f_{DEX}))} > 0$ for all $f_{DEX} \in (\hat{f}_-, f_{CEX})$.

□

B.5 Proof of Proposition 3.3

Proof. We first note that

$$P^{\star}_{DEX}(\delta) = \frac{VI^{\star}(f)}{I^{\star}(f) - 2V\delta}$$

and therefore

$$\lambda^{\star}(f, \delta) = 2V^2 \cdot \frac{I^{\star}(f)}{(I^{\star}(f) - 2V\delta)^2}$$

Next, note that

$$\frac{d}{df}\lambda^{\star}(f, \delta) = -2V^2 \cdot \frac{I^{\star}(f) + 2V\delta}{(I^{\star}(f) - 2V\delta)^3} \cdot \frac{\partial I^{\star}(f)}{\partial f}$$

Finally, we note that $\frac{I^{\star}(f)+2V\delta}{(I^{\star}(f)-2V\delta)^3} > 0$ for all feasible trades as $\delta < I^{\star}_{ETH}$ implies $2V\delta < I^{\star}(f_{DEX})$. Further, we have shown in the proof of Proposition 3.3 that

$$\frac{\partial^2}{\partial f^2} log(I^{\star}(f)) < 0$$

and therefore $I^{\star}(f)$ has a unique local maximum, which combined with the fact that $I^{\star}(f) \geqslant 0$ for all $f \in [0, f_{CEX}]$ and $I^{\star}(0) = I^{\star}(f_{CEX}) = 0$, implies that there exists $\tilde{f} \in (0, f_{CEX})$ such that $\frac{\partial I^{\star}(f)}{\partial f} > 0$ whenever $f < \tilde{f}$ and $\frac{\partial I^{\star}(f)}{\partial f} < 0$ whenever $f > \tilde{f}$. Hence, $\frac{d}{df}\lambda^{\star}(f, \delta) < 0$ for all $f \in (0, \tilde{f})$ and $\frac{d}{df}\lambda^{\star}(f, \delta) > 0$ for all $f \in (\tilde{f}, f_{CEX})$. □

B.6 Proof of Proposition 3.4

Proof. First, we note that

$$\lambda^{\star}(I, \delta) = 2V^2 \frac{I}{I - 2V\delta}$$

thus,

$$\frac{d\lambda^{\star}(I, \delta)}{dI} = -2V^2 \frac{I + 2V\delta}{(I - 2V\delta)^3} < 0$$

for all possible inventory levels I and feasible trades $\delta < I_{ETH} = \frac{1}{2V} I$. □

B.7 Proof of Proposition 3.5

Proof. We have shown in the proof of Proposition 3.2 that $\frac{\partial^2}{\partial f^2} log(I^{\star}(f)) < 0$ for all $f \in (0, f_{CEX})$ and therefore there is a unique critical point of $I^{\star}(f)$ over the interval $(0, f_{CEX})$. Combining this with the fact that $I^\star(f) > 0$ for all $f \in (0, f_{CEX})$ and $I^\star(0) = I^\star(f_{CEX}) = 0$ then implies that there must exists $\tilde{f}$ such that $f \in (0, \tilde{f})$ implies $\frac{\partial I^\star}{\partial f_{DEX}} > 0$ for $f_{DEX} \in (0, \tilde{f})$ and $\frac{\partial I^\star}{\partial f_{DEX}} < 0$ for all $f_{DEX} \in (\tilde{f}, f_{CEX})$. □

B.8 Proof of Proposition 3.6

Proof. In order to prove this result, we simply note that $r^\star_{DEX}(f_{DEX}) = I^\star(f_{DEX})^\theta$. Therefore, $\frac{dr^\star_{DEX}}{df_{DEX}} = \theta(I^\star(f_{DEX}))^{\theta-1}\frac{dI^\star}{df_{DEX}}$ and we know from Proposition 3.5 that there exists $\tilde{f} \in (0, f_{CEX})$ such that $\frac{dI^\star}{df_{DEX}} > 0$ when $f \in (0, \tilde{f})$ while $\frac{dI^\star}{df_{DEX}} < 0$ when $f \in (\tilde{f}, f_{CEX})$. Therefore, given that $I^\star(f) > 0$ for all $f \in (0, f_{CEX})$ it must be the case that $\frac{dr^\star_{DEX}}{df_{DEX}} > 0$ for all $f \in (0, \tilde{f})$ and $\frac{dr^\star_{DEX}}{df_{DEX}} < 0$ for all $f \in (\tilde{f}, f_{CEX})$. □