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Is 24/7 Trading Better?

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ABSTRACT

Are daily market closures still needed? In a model of large traders who manage inventory risk, we show that even short market closures can significantly improve liquidity. Anticipating these closures, traders engage in aggressive trading, which concentrates and coordinates liquidity. A market structure with a daily closure improves allocative efficiency relative to a continuously open market, even though traders cannot trade during the closure itself. If traders have heterogeneous information about the asset value, trade is less aggressive on the whole, but closure still retains its substantial welfare benefits. Our findings suggest moving to a longer trading day could be beneficial, but moving to 24/7 trading would harm welfare.

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I. Introduction

Since the founding of the NYSE in 1792, trading hours have closely mirrored the conventional workday due to the human involvement that was essential for trading. In the 21st century, electronic execution has become the dominant form of trade for standardized products.[1] This transition has helped facilitate faster and more efficient trade (Litzenberger et al., 2012), with little to no human interaction. Beyond the electronic execution of trades, the majority of trades are submitted by algorithmic traders.[2] Given electronic trading and execution have substantially reduced the need for constant human involvement, market participants and regulators are actively discussing whether markets should be open 24/7.[3] This paper studies the implications of market closures for welfare.[4]

We study a model of large traders who rationally anticipate how their demand affects prices while managing risky inventory positions. Traders experience inventory shocks that move their inventories away from their desired positions, motivating trade. We study welfare, measured by the allocative efficiency of the market, in equilibria of two market designs: one in which there is a daily closure and another in which closure is eliminated. A daily closure is costly because it eliminates traders' ability to manage their inventory for a fraction of the day, leading traders to arrive at the start of the next day in positions that may be far from desirable. Is there a benefit to daily market closures?

We show that a market design with a daily market closure of some length is strictly better than having trade occur 24/7. Traders rationally anticipate being unable to directly manage their inventory positions overnight, which incentivizes them to be in a good inventory position by the end of the trading day. Therefore, all traders trade more aggressively towards a desirable position throughout the day, particularly at the end of the day. In turn, this aggressive trade increases

[1]Out of thirteen registered equity exchanges, the NYSE is the only that is not 100% electronic.

[2]In 2003, about 15% of the U.S. equity volume consisted of algorithmic trades, whereas in 2023, estimates suggest $70-80\%$ of U.S. equity volume is algorithmic trades (SEC, 2020). Brogaard et al. (2024) show that floor brokers still play an important role in trading on the NYSE, though most of their benefits are concentrated at the open, the close, and early in the trading day. Their role could easily still be preserved in an extended trading hour environment by having the closure, for example, be between 2:00-3:00 PM EST.

[3]The NYSE polled market participants about 24/7 trading in April 2024 (`https://www.ft.com/content/31c3a55b-9af9-4158-8a49-4397540571bf`). 24X, a part of Steve Cohen's Point72 Venture fund, has filed for a second time for the SEC's approval to launch the first 24/7 equity exchange (`https://www.sec.gov/files/rules/other/2024/34-100254.pdf`). Robinhood and Interactive Brokers already allow trade 24/7 for some stocks and ETFs by internalizing the trades.

[4]While a natural application of the theory developed in this paper is equities and the NYSE, our results are applicable to other asset classes and exchanges.

liquidity throughout the day, which lowers the cost of trade and further incentivizes aggressive trade as the market closure approaches. Therefore, "liquidity begets liquidity." The market closure coordinates and concentrates liquidity, reducing the amount of undesired inventory held across traders, especially at the end of the day. Contrary to the basic intuition that a constraint on trade is purely a cost, the endogenous coordination of trade due to market closure can more than offset this cost.

When trade is 24/7, there is no equilibrium in which traders coordinate trade. Since traders rationally anticipate how their demand affects prices and future inventory positions, they break up their orders over time to minimize execution costs, which creates socially inefficient excess inventory costs (Du and Zhu, 2017, Rostek and Weretka, 2015, Vayanos, 1999). As liquidity is spread out, price impact further increases, which further incentivizes traders to break up their orders. In this market design, liquidity, a public good, is spread very thinly throughout the trading day. A market closure can benefit by concentrating and coordinating liquidity.[5]

Having established introducing some length of closure would benefit a market with 24/7 trading, we then study how long the optimal closure should be. Empirically, this portion of the paper points to the heterogeneous cross-section of trading day lengths across exchanges and security types.[6] Increasing the length of closure magnifies the concentration of liquidity during the trading day at the expense of reducing traders' ability to manage their inventory position through a longer overnight period. In general, we find that the costs of increasing the length of closure are strong, as any nonzero length of daily market closure provides coordination of liquidity. Thus, a benevolent social planner maximizing the ex-ante welfare of the traders would choose a short length of market closure. Further, this result is under the assumption that the volatility of the shocks to a trader's inventory position is the same overnight as during the trading day. If the volatility of shocks (and/or holding costs) during the trading day is greater than overnight, then the costs of daily market closure decrease, causing a longer market closure to become optimal. This result is particularly strong in

[5] Even in securities that trade 24/7, such as forex, we empirically observe volume spikes coinciding with the opening and closing of other exchanges, such as the NYSE. These volume spikes suggest market closures are even important across asset exchanges.

[6] The NYSE and Nasdaq operate from 9:30 AM - 4:00 PM EST (6.5 hours). The CBOE operates from 9:30 AM - 4:00 PM EST (6.5 hours). The CME Globex operates 6:00 PM - 5:00 PM EST (23 hours). MarketAxess' SEF operates 2:00 AM - 5:30 PM EST (15.5 hours). GFI operates 7:00 AM - 5:00 PM EST (10 hours). Finally, most forex and cryptocurrency exchanges operate 24 hours a day, while some US Treasuries and stock index futures trade 24/5.

small markets, where liquidity is already fairly thin, and the liquidity concentration benefits of a closure are substantial.

To study the magnitude of the benefits market closure provides, we analyze welfare under various market designs relative to first-best (efficient) market allocations, which avoid the social costs of strategic trade. We find that 24/7 trading is worse by a factor of two relative to the welfare achieved from efficient allocations for all market sizes. For example, the common market design of trading for 6.5 hours a day for equities is more efficient than 24/7 trading when the market is small, but the long closure becomes very costly in larger, more liquid, markets. Finally, a market closure close to the optimal length drastically reduces the costs of imperfect competition relative to a market structure with 24/7 trading.

Lastly, we analyze the extent to which the model with market closures captures prominent intraday patterns in financial markets. As in the data (e.g., Chan et al. (1996), Jain and Joh (1988)), intraday volume is U-shaped. Trade in the model can be decomposed into two components, which vary over time: a component that determines the gap a trader faces between their current and desired inventory and a component that determines how aggressively a trader trades to eliminate this gap. At the start of the day, traders face large gaps between their current and desired inventory, resulting in a large volume, even though they trade away this gap slowly. At the end of the day, traders trade very aggressively. So, even though the gap between current and desired is not very large, this aggressive trade results in large volume. In the middle of the day, traders' gaps between desired and current inventories are not very large, and trade is not particularly aggressive, resulting in low volume relative to other parts of the day.

The model can also capture stylized facts about returns throughout the day. In particular, if the traders in the model face inventory shocks that force them to sell the asset on average, intraday returns are, on average, negative while overnight returns are positive (e.g., Lu et al. (2023), Branch and Ma (2012), Kelly and Clark (2011), Cliff et al. (2008), Branch and Ma (2006)). This result can be understood when one considers how aggregate risk-bearing capacity in the model behaves over the trading day. At the end of the trading day, traders reach the desired socially optimal allocations, and the risk-bearing capacity is maximized. Earlier in the day, risk-bearing capacity is low as traders' inventories deviate from socially optimal allocations. Therefore, a risk premium is earned throughout the day as risk-bearing capacity improves. When traders face net buying pressure, this risk-bearing capacity manifests itself in terms of a negative intraday return.

We also show that this paper's main results are robust to including heterogeneity in beliefs about asset values through noisy private signals. An interesting implication of this extension is that heterogeneity tends to reduce the aggressiveness of trade overall. Yet, closure still concentrates liquidity, allowing traders to trade very aggressively at the end of the day with minimal price impact and improving welfare.

There is an extensive literature studying financial markets both intraday and overnight. Patterns related to market closures that have been seen in the data include U-shaped intraday mean returns and volatility [7], U-shaped intraday volume[8], open-to-open returns being more volatile than close-to-close returns[9], returns over trading periods being more volatile than returns over non-trading periods[10], returns being higher overnight than intraday[11], and intraday and overnight variation in factor pricing[12]. These patterns are not exclusive to equities and can also be found in other asset classes. There is also substantial literature that theoretically explains these facts (Hong and Wang, 2000, Slezak, 1994, Foster and Viswanathan, 1993, Brock and Kleidon, 1992, Foster and Viswanathan, 1990, Admati and Pfleiderer, 1989, 1988). All of this literature takes the existence and length of a nighttime as fixed. This paper studies variations in the existence and length of a nighttime.

This paper also contributes to the literature that studies how common financial market structures interact with strategic trading and their implications for the allocative efficiency of the market. Chen and Duffie (2021), Malamud and Rostek (2017), and Kawakami (2017) study market fragmentation. Fuchs and Skrzypacz (2019), Du and Zhu (2017) and Vayanos (1999) study trading frequency. Blonien (2024), Antill and Duffie (2020), and Duffie and Zhu (2017) examine the addition of a trading session at a fixed price. Fuchs and Skrzypacz (2015) study government market freezes in a dynamic adverse selection model. deHaan and Glover (2024) provide empirical evidence that retail traders achieve better portfolio performance when trading hours decrease. Apart from deHaan and Glover (2024), whose focus is on retail trade, none of these papers study the effects of

[7] For example, Andersen and Bollerslev (1997), Harris (1989, 1988, 1986).

[8] For example, Chan et al. (1996), Jain and Joh (1988).

[9] For example, Amihud and Mendelson (1991), Stoll and Whaley (1990), Amihud and Mendelson (1987).

[10] For example, Amihud and Mendelson (1991), Barclay et al. (1990), French and Roll (1986), Fama (1965).

[11] For example, Branch and Ma (2012), Kelly and Clark (2011), Cliff et al. (2008), Branch and Ma (2006).

[12] For example, Bogousslavsky (2021), Hendershott et al. (2020), Lou et al. (2019).

daily market closures.

The presence of market closures is closely linked to the existence of closing auctions, whose characteristics have been of recent interest. The percentage of daily volume transacted in these special sessions has reached an all-time high in recent years (Bogousslavsky and Muravyev, 2023). Consistent with this, our model generates a substantial fraction of daily volume near the open and close. The Autorité des Marchés Financiers (AMF, 2019) has expressed concerns that this increase in concentration at the close may lead to price and liquidity deterioration during trading beforehand. An important result is that we find prices are closer to fundamental value due to the existence of a closure, as the premium for holding inventory is smaller. We also find that a market closure generates liquidity and volume throughout the trading day, not simply at the close. Bogousslavsky and Muravyev (2023), Jegadeesh and Wu (2022), and Hu and Murphy (2022) compare the NYSE and Nasdaq closing auctions to study liquidity and price efficiency around the closing auction. This paper speaks to what would happen to volumes and prices at the open and close if the length of the trading day was altered.

The paper proceeds as follows. Section II defines the model. Section III defines and solves for the equilibrium and builds intuition for how the traders optimally trade with and without a market closure. Section IV quantifies welfare and our main results. Section V studies which intraday patterns our model matches. Section VI extends the model to allow for heterogeneous information. Section VII concludes. The Appendices provide technical details and proofs.

II. The Model

Section II.A introduces a model of strategic trading under imperfect competition with periodic market closures. Section III.A introduces a version of the model without market closure that is equivalent to special cases of the models studied in Antill and Duffie (2020), Du and Zhu (2017).

A. A Model with a Nighttime

Time goes from 0 to ∞. We set a unit of clock time to be 24 hours. We refer to the fraction of the 24 hours where trade occurs as "day" and the remaining fraction, where no trade is permitted, is referred to as "night". Let $\Delta \in [0, 1)$ be the length of the night. In the first day, trade occurs from time 0 to $1-\Delta$, and night occurs from $1-\Delta$ to 1. Each unit of clock time repeats under this format forever.

There are $N \geq 3$ risk-neutral traders who trade a divisible asset. Traders want to hold the asset because it pays a liquidating dividend of v per unit of inventory held. The time of liquidation is random and exponentially distributed, denoted $\mathcal{T} \sim Exp(\lambda)$, so that the expected time until liquidation is $\frac{1}{\lambda}$.[13][14]

The frequency of trade is continuous. Each trading session of length dt is modelled as a uniform-price double auction. During trade at date t, each trader i submits a demand schedule $D_t^i : \mathbb{R} \to \mathbb{R}$ that is a mapping of price to demand, $p \mapsto D_t^i(p)$. The market clearing price, p_t^*, is the price that sets net demand to be zero,

$$\sum_{i=1}^{N} D_t^i(p_t^*) = 0. \tag{1}$$

Each trader then pays the equilibrium price, p_t^*, times the amount of the asset they were allocated, $D_t^i(p_t^*)dt$. If $D_t^i(p_t^*)dt < 0$, then trader i receives the equilibrium price times the amount of the asset they were allocated. The modeling of trade as a match auction as opposed to a limit-order book provides tractability while maintaining the important economic mechanism of price impact from trade.[15]

Each trader is endowed with some portion of the asset, referred to as the trader's initial inventory. Inventories are subject to exogenous shocks, and can be managed endogenously through trade. In fact, it is the exogenous shocks to inventory positions that induce trade. Trader i's inventory at time t is denoted z_t^i, so that inventory evolves according to

$$dz_t^i = D_t^i dt + \sigma_d dB_t^i, \tag{2}$$

during the day and according to

$$dz_t^i = \sigma_n dB_t^i \tag{3}$$

during the night. B_t^i is a standard Brownian motion, and $\sigma_d, \sigma_n \in \mathbb{R}$ are not necessarily equal. B_t^i can have an arbitrary correlation structure across traders. As these expressions make clear, traders

[13] Allowing the dividend to be paid continuously with common discount rate λ leads to an essentially identical model and results. Also, the dividend is liquidating for simplicity. One could also allow discrete dividends to be paid at a rate forever.

[14] The main result, Proposition 3, is robust to having the probability of liquidation set to zero during the night.

[15] A series of papers (Budish et al., 2024, Aquilina et al., 2021, Budish et al., 2015) recommend that the market be switched from a continuous limit-order book to frequent batch auctions to curtail the socially wasteful arms race for speed.

can only manage inventory through trade during the day, although inventory is always subject to exogenous shocks.

Finally, each trader incurs a holding cost per unit time of $\gamma_d \times (z_t^i)^2$ during the day and $\gamma_n \times (z_t^i)^2$ during the night. Blonien (2024), Chen (2022), Chen and Duffie (2021), Antill and Duffie (2020), Duffie and Zhu (2017), Du and Zhu (2017), Sannikov and Skrzypacz (2016), Rostek and Weretka (2012) and Vives (2011) all use a similar quadratic holding cost. This cost can be interpreted as a representing inventory costs, collateral requirements, or risk management. Since the cost is convex, traders are averse to large movements in their inventory, and their inventory management amounts to reducing the impact of exogenous shocks to inventory through trade.

Now let us define the traders' value functions. In the following sections, we will study equilibria that are periodic, with period equal to one day. Therefore, to ease the exposition, we simply focus on $t \in [0, 1]$, and note results at any other time are analogous. Recall the first trading day is from time 0 to $1 - \Delta$. So, for $t \in [0, 1 - \Delta)$, define a trader's value function J^d, which is a function of time t, current inventory position z^i, and aggregate inventory $\bar{Z} = \frac{1}{N} \sum_{i=1}^{N} z^i$, by

$$J^d(t, z^i, \bar{Z}) = \sup_{\{D_s^i\}_{s=t}^{\infty}} \mathbb{E}_t \Bigg[\int_t^{\min(\mathcal{T}, 1-\Delta)} (- \underbrace{p_s^* D_s^i}_{\substack{\text{cost of} \\ \text{allocation}}} - \underbrace{\gamma_d (z_s^i)^2}_{\substack{\text{inventory} \\ \text{flow cost}}}) ds + \underbrace{\mathbb{1}_{\mathcal{T} \leq 1-\Delta}}_{\substack{\text{liquidation} \\ \text{during day}}} \underbrace{z_{\mathcal{T}}^i v}_{\substack{\text{liquidation} \\ \text{value}}} + \underbrace{\mathbb{1}_{\mathcal{T} > 1-\Delta}}_{\substack{\text{no liquidation} \\ \text{during day}}} \underbrace{J^n(1 - \Delta, z^i, \bar{Z})}_{\substack{\text{value function} \\ \text{at start of night}}} \Bigg]. \tag{4}$$

The maximum is over demand schedules, not simply realized demands. In equilibrium, prices reveal aggregate inventory $\bar{Z}$. Therefore, the value function J^d in the day is a function of $\bar{Z}$ insofar as it affects future prices and realized demands, and thus utility. Yet without trade, $\bar{Z}$ is not directly observed, and its conditional expectation $\hat{Z}_t \equiv \mathbb{E}_t[Z|t]$ becomes a state variable. Traders simply infer Z from their own inventory, to the extent that shocks to the two are correlated. Then, for $t \in [1 - \Delta, 1)$, define the value function J^n, which is a function of time t, the current inventory position z^i, and the expected aggregate inventory $\hat{Z}$, by

$$J^n(t, z^i, \hat{Z}) = \mathbb{E}_t \Bigg[\int_t^{\min(\mathcal{T}, 1)} (- \underbrace{\gamma_n (z_s^i)^2}_{\substack{\text{inventory} \\ \text{flow cost}}}) ds + \underbrace{\mathbb{1}_{\mathcal{T} \leq 1}}_{\substack{\text{liquidation} \\ \text{at night}}} \underbrace{z_{\mathcal{T}}^i v}_{\substack{\text{liquidation} \\ \text{value} \\ \times\text{number of shares} \\ \times\text{value per share}}} + \underbrace{\mathbb{1}_{\mathcal{T} > 1}}_{\substack{\text{no liquidation} \\ \text{at night}}} \underbrace{J^d(1, z^i, Z)}_{\substack{\text{value function} \\ \text{at start of day}}} \Bigg]. \tag{5}$$

These value functions are subject to inventories evolving as in Equations (2) and (3).

Let's summarize the terms in these value functions. Upon liquidation, each trader receives the value of their position and the economy ends. Before liquidation, during the day, each trader pays incurs the flow costs of their trades, $p_s^* D_s^i$, and the quadratic flow costs of their inventory positions, $\gamma_d (z_s^i)^2$. At night, each trader incurs the flow costs of their inventory positions, $\gamma_n (z_s^i)^2$.

III. Equilibrium

Prior literature (e.g., Antill and Duffie (2020), Du and Zhu (2017)) frequently studies symmetric, linear, and stationary equilibria. That is, the equilibrium demand schedules of each trader are the same linear combination of price and other relevant inputs. In our model with daily market closures, such an equilibrium will not exist. The trading problem that faces every trader will not be ex-ante identical at each trading session, as the opportunity set changes throughout the day, precluding the existence of stationary equilibria. For instance, as the closure approaches, traders will behave differently as the inability to manage inventory overnight presents a substantial change in the opportunity set.

Therefore, we focus on symmetric, linear, and periodic demand schedules. For example, in equilibrium, all demand schedules submitted at 9:30 AM will be the same every day, but all traders may use a different demand schedule at 10:00 AM than they did at 9:30 AM. Concretely, we conjecture that the equilibrium demand schedule is of the following form:

$$D^i(t, z^i, p) = a(t) + b(t)p + c(t)z^i. \tag{6}$$

By market clearing, trader i faces the residual supply curve of the other $N-1$ traders and chooses a price and quantity pair. If trader i chooses demand quantity d^i, then by market clearing, the price must solve $d^i + \sum_{j \neq i} (a(t) + b(t)p + c(t)z_t^j) = 0$. Therefore, the market clearing price is

$$\Phi(t, d^i, Z^{-i}) := p = -\frac{1}{b(t)(N-1)} \left(d^i + (N-1)a(t) + c(t)Z^{-i} \right), \tag{7}$$

where $Z^{-i} = \sum_{i \neq j} z^j$. Thus, traders are strategic in that they rationally anticipate and internalize how their demand affects prices due to imperfect competition. As price impact itself is only a wealth transfer between traders, it is the strategic effects of avoiding price impact that can be socially costly by reducing allocative efficiency.

Having discussed traders' decision sets, let us clarify the impact of periodicity on the value functions. Periodicity requires that J^d, J^n are periodic functions of time, with period 1. Further, there are two boundary conditions that the value function must obey in equilibrium. We simply state them in the first trading day for simplicity. The day value function at the end of the trading day should equal the night value function at the start of night, $J^d(t = 1 - \Delta, z^i, \bar{Z}) = J^n(t = 1 - \Delta, z^i, \bar{Z})$. Second, the night value function at the end of the night should equal the expected day value function at the start of the day, conditional on the information right before the first trading session of the day, $J^n(1, z^i, \hat{Z}) = \lim_{t \to 1^-} \mathbb{E}\left[J^d(t = 0, z^i, \bar{Z}) \middle| \mathcal{I}_t\right]$.

A symmetric equilibrium of the above stochastic game is defined by the functions $a(t)$, $b(t)$, and $c(t)$. Equilibrium requires that, if trader i conjectures the other $N-1$ traders use the linear demand schedule (6), trader i's best response is to use the same demand schedule, and the market clears. We show below in the Appendix that this equilibrium exists and is characterized by Proposition 1.

PROPOSITION 1: *There exists an unique symmetric, linear, and periodic equilibrium with the following properties for $t \in [0, 1 - \Delta)$:*

1. *The optimal equilibrium demand schedule every trader plays is*
$$D^i(t, z^i, p) = \frac{\lambda(N-2)}{2\big(1 - e^{-\lambda(1-\Delta-t)}\big)} \left(\frac{\lambda}{2\gamma_d\big(1 - e^{-\lambda(1-\Delta-t)}\big)} (v - p) - z^i_t \right). \tag{8}$$
2. *The equilibrium market clearing price is*
$$p^*_t = v - \frac{2\gamma_d}{\lambda}\big(1 - e^{-\lambda(1-\Delta-t)}\big)\bar{Z}_t. \tag{9}$$
3. *Therefore, the equilibrium allocation per trader is*
$$D^i(t, z^i, p^*) = -\frac{\lambda(N-2)}{2\big(1 - e^{-\lambda(1-\Delta-t)}\big)} \left(z^i_t - \bar{Z}_t \right). \tag{10}$$

Let us discuss these results. The coefficient on $z_i - \bar{Z}_t$ in the equilibrium allocation diverges as the last trading session of the trading day approaches. This causes the gap between desired and current inventory positions, $z^i - \bar{Z}$, to completely close as trading day ends. Essentially, this fact renders the excess inventory process $z^i - \bar{Z}$ akin to a Brownian bridge, starting the trading day at $z^i_0 - \bar{Z}_0$, and ending at 0. Let us explain the intuition driving this result. As the end of day approaches, traders are aware that they will lose the opportunity to trade to share their own inventory risk and will be unable to manage the random shocks to their inventory. They all, therefore, have the incentive to enter nighttime in the best possible inventory position. As a result, traders are more willing to incur price impact and temporary trading costs towards the end of the trading day. The old adage of "liquidity begets liquidity" comes into effect; more liquidity is supplied due to the fear of excess inventory costs overnight, so it becomes even cheaper to trade more aggressively now, further encouraging aggressive trade now.

As the excess inventory is driven to zero at the end of the trading day, the equilibrium price converges to the expected value of the dividend going forward, v. Before the end of the trading day, though, there is a compensation term due to aggregate inventory holding costs. This fact is discussed further below when we discuss intraday return patterns. Another implication of the excess inventory for all traders going to zero at the end of the trading day is that the costs of inventory overnight, γ_n, or volatility of the shocks to their excess inventory, σ_n, does not affect their trading strategies during the trading day, surprisingly. It does affect their value functions and, therefore, the welfare statements we will make later, however.

A. Model of 24/7 *Trading*

As we show in this section, the solution of the model above with no closure is not simply the limit of the solution characterized in Proposition 1 as the length of night Δ goes to 0. Therefore, in this section, we study how market participants behave when there is never a market closure, which we refer to as 24/7 trading.

The model with no nighttime, $\Delta = 0$, is a special case of Du and Zhu (2017), where the time between trades goes to zero and there is no adverse selection, and Antill and Duffie (2020), when the rate of occurrence of a size-discovery session occurs is set to zero. We make no other modifications to the model from the previous section other than setting $\Delta = 0$. Once again, we differ from most prior literature by conjecturing linear, symmetric, and periodic equilibria of the same form as Equation 6. Periodicity again requires the value function J is a periodic function with period 1. Further, the value function must be continuous from the end of one day to the start of the next, $J(t = n, z^i, \bar{Z}) = J(t = n + 1, z^i, \bar{Z}), n \in \mathbb{N}$.

We characterize the equilibrium in Proposition 2.

PROPOSITION 2: *There exists an unique symmetric, linear, and periodic equilibrium with the following properties:*

1. *The optimal equilibrium demand schedule every trader plays is*

$$D^i(t, z^i, p) = \frac{\lambda(N-2)}{2}\left(\frac{\lambda}{2\gamma_d}(v-p) - z_t^i\right). \tag{11}$$

2. *The equilibrium market clearing price is*

$$p_t^* = v - \frac{2\gamma_d}{\lambda}\bar{Z}_t. \tag{12}$$

3. *The equilibrium allocation per trader is*

$$D^i(t, z^i, p^*) = -\frac{\lambda(N-2)}{2}\left(z_t^i - \bar{Z}_t\right). \tag{13}$$

A first point to note is that the equilibrium strategy played is time-invariant. Despite allowing the demand schedules submitted to be periodic across days, the unique equilibrium is constant across time, as in Antill and Duffie (2020) and Du and Zhu (2017). Therefore, conditional on state variables $z^i, \bar{Z}$, prices, and allocations are the same across time.

Next, we compare the equilibrium in Proposition 1, when there is a market closure, and Proposition 2, where trading occurs 24/7. The first thing to note is how similar the equilibria in Proposition 1 and Proposition 2 look. The only difference is a $1 - e^{-\lambda(1-\Delta-t)}$ term that appears several times. Note that for any time during the trading day, $t \in [0, 1-\Delta)$, $1 - e^{-\lambda(1-\Delta-t)} < 1$. Therefore, when comparing Proposition 1 and Proposition 2, we see that because of night, the premium for holding inventory is always less than the model with 24/7 trading. This result is due to the fact that you only expect to potentially hold unwanted inventory until the end of the trading day. Also, because of the coordination and concentration of liquidity in the lead-up to the closure, the equilibrium allocation is always strictly more aggressive during the trading day when there is a market closure.

Figure 1 quantifies the magnitude of the coordination and concentration of liquidity when there is a market closure. The y-axis is the expected percentage of excess inventory left for a given trader relative to the start of the day. When there is not a closure, note that c is constant, but $c(t)$ becoming increasingly negative throughout the trading day when there is a closure. In general, the gaps between the same colors, which correspond to comparisons between different market designs for a fixed market size, are very large. When the market is small, $N = 10$, and there is no closure, liquidity is spread so thinly through the day that very little undesired inventory is offloaded, about 20%, even after a full 24 hours of trading. When there is a market closure for just 10% of the day,

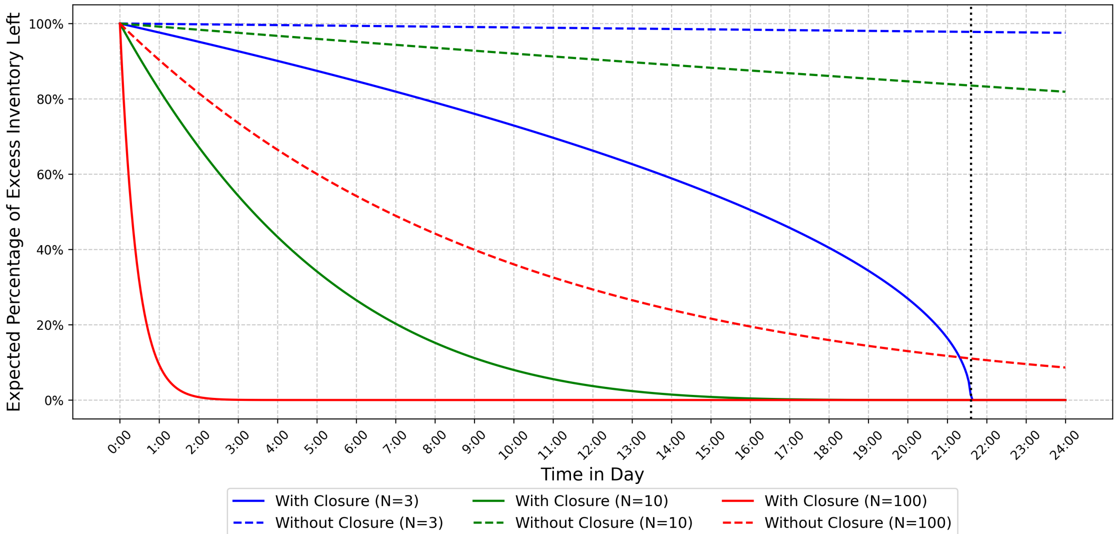


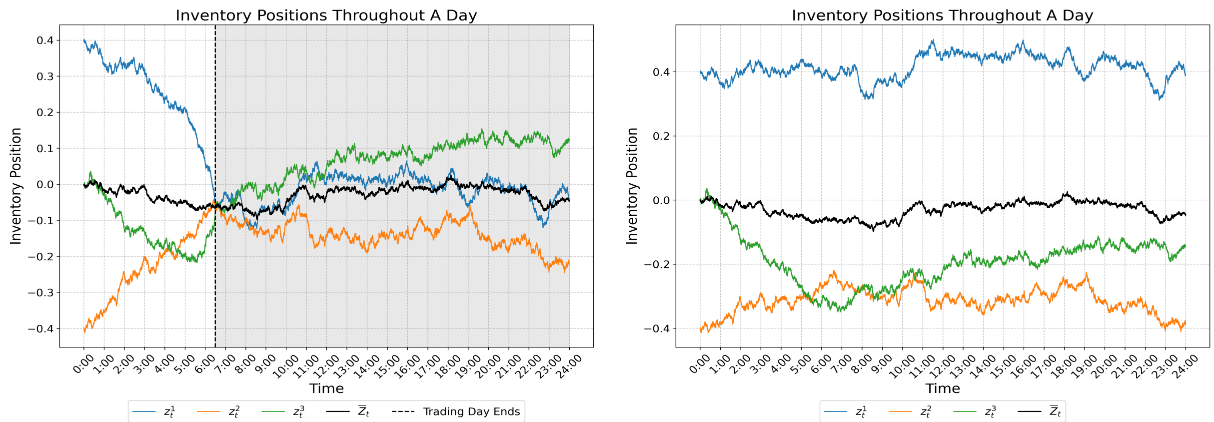
Figure 1. This figure plots trading intensity for various different regimes throughout the day. $\exp(\int_0^t c(s)ds)$ is the expression for the expected percentage of excess inventory left at point t in the day relative to the position at the start of the day for a given market design. The solid lines are from Proposition 1, and the dashed lines are from Proposition 2. The colors map to the size of the market, with blue being $N = 3$, green being $N = 10$, and red being $N = 100$. The vertical dotted line is when the market closes for trading with respect to Proposition 1. We set $\lambda = 5\%$ and $\Delta = 10\%$.

traders have already unloaded about 20% of their undesired inventory in just an hour. This plot speaks to how strong the benefits of coordinating liquidity can be.

B. Simulating the Models

In this subsection, we simulate a trading day for a market with three traders. We conduct this simulation in two scenarios: first, when trade only occurs for the first 6.5 hours of the day, followed by nighttime and no trade for 17.5 hours, and, second, trade can occur 24/7. Each trader receives the same shocks to their inventory position in the two scenarios. We set $N = 3$, $\lambda = 5\%$, $\sigma_d = \sigma_n = 20\%$, and the initial inventory positions to be $z_0^1 = .4$, $z_0^2 = -.4$, and $z_0^3 = 0$, which implies $\bar{Z}_0 = 0$. Since the shocks to each trader's inventory position are the same across the two plots, the average aggregate inventory outstanding, which the traders are attempting to reach, is the same in each scenario, as its drift is zero by market clearing.

The results of these simulations are plotted in Figure 2. Starting with Figure 2(a), while there is noise in the traders' inventory positions during the trading day, all of their inventories converge



(a) Simulation of model with $\Delta = 17.5$ hours of night

(b) Simulation with trading 24/7 ($\Delta = 0$)

Figure 2. These figures plot inventory paths under the same simulated shocks over a single day for three traders, $N = 3$, but the left plot has a nighttime of 17.5 hours, and the right plot has trading 24/7. The average aggregate inventory (the solid black line), $\bar{Z}_t$, is the same across both plots. The parameters used are $\sigma_d = \sigma_n = 20\%$ and $\lambda = 5\%$. The optimal length of the night in this example would be $\Delta^* = 7\%$.

exactly to their desired perfect risk-sharing inventory position, $\bar{Z}$, at the end of the trading day. While they slowly trade towards it earlier in the day, they become very aggressive in the last hour or so. After the trading day ends, traders can no longer control the drift of their inventory positions. Therefore, inventories randomly evolve overnight. Traders dislike potentially incurring excess flow costs overnight due to the random shocks to their inventory position. Yet, traders potentially incurred much less flow cost during the day and early at night due to the ability to trade cheaply toward their desired inventory positions. Whether this is net beneficial or not will be studied formally in Proposition 3.

In Figure 2(b), there is no market closure. Without market closure, traders strategically break up their orders over time, spreading out liquidity and trading slowly toward their desired inventory position $\bar{Z}$. Without the coordination and concentration of liquidity, trader never substantially close the gap. This single simulation shows the amount of excess inventory held over the 24-hour period can be clearly higher when trading occurs for all 24 hours. Yet, with only three traders, liquidity is very low, and therefore, the benefits of coordinating and concentrating liquidity are very large. As we will see below, a market closure of this length will not always be better than 24/7 trading, but there will always exist some length of a closure that will be improve on a market structure with 24/7 trading.

IV. Welfare

We define welfare as the sum of the ex-ante expected value of the value functions across all traders in the market. As each trader's value function aggregates their expected profits net of flow costs, the higher the value, the more efficient the market is. In this section, we assume that the initial inventory position for each trader is $N(0, \sigma_d^2)$ distributed. As a first benchmark, we define the first-best (efficient) welfare as that which continuously and perfectly reallocates each trader's inventory position to the perfect risk-sharing amount. This benchmark is what a benevolent social planner would achieve if we eliminated both frictions in the model by making markets perfectly competitive and letting trade occur 24/7. The efficient welfare is

$$W^e := \sum_{i=1}^{N} \mathbb{E}\big[J^e(\bar{Z})\big] = -\frac{\gamma_d \sigma_d^2 (1+\lambda)}{\lambda^2}. \tag{14}$$

Next, we quantify welfare under the market design that never has a market closure where trade occurs continuously 24/7. This is the welfare from Proposition 2. This 24/7 welfare is

$$W^{24/7} := \sum_{i=1}^{N} \mathbb{E}\big[J(z^i, \bar{Z})\big] = -\frac{2\gamma_d \sigma_d^2 (1+\lambda)}{\lambda^2}. \tag{15}$$

Finally, we quantify the welfare achieved from Proposition 1, where there is a market closure of length $\Delta \in (0, 1)$. While we have solved the model more generally and can write down welfare for that model, for simplicity, we now assume that $\sigma_d = \sigma_n$, $\gamma_d = \gamma_n$, and the Brownian shocks are independent across traders. Then, welfare under a market closure of length Δ is

$$W(\Delta) := \sum_{i=1}^{N} \mathbb{E}\left[J^d(0, z^i, \bar{Z})\right] =$$
$$\frac{\gamma_d \sigma_d^2}{\lambda}\left(\frac{-2(1+\lambda)}{\lambda} + \frac{(1+\lambda\Delta)(N-2) + e^{-\lambda(1-\Delta)}\big(\lambda\Delta + e^{\lambda}(2 - N + \lambda(1-\Delta))\big)}{\lambda(e^{\lambda}-1)} + e^{-\lambda(1-\Delta)}\right). \tag{16}$$

First, note the flow cost and volatility of the inventory shock parameters positively welfare under each market structure. Therefore, we will be able to make fairly general statements without having to take a stand on their values as long as they are weakly positive. Next, note the maximal ex-ante welfare achievable, W^e, is negative, and, therefore, all ex-ante welfares will be negative.[16] The fact that welfare is negative is due to the fact that, unconditionally, the asset supply has a mean of zero. Even though the asset's value could be very positive, the market is equally short

[16] One could, of course, easily add a positive constant to each trader's flow utility, and thus value function, or make the asset not in zero net supply unconditionally.

as it is long. Due to market clearing, the costs of allocated equilibrium trades also net out to zero since trades are merely wealth transfers. Yet, in addition to these two 0 terms, there is a strictly negative flow cost, which drives welfare to be strictly negative. The fact that welfare is strictly negative does not violate an individual rationality condition either. When solving for the equilibrium, submitting a demanding schedule of zero is an allowable strategy, but it is optimal for each trader to participate in every trading session. Last, note the cost of imperfect competition and traders internalizing their price impacts lowers welfare by a factor of 2, as $\frac{W^{24/7}}{2} = W^e < 0$. The relationship between $W(\Delta)$ and the other two welfare expression is much less clear. It will depend on the number of traders in the market, N, which does not appear in the efficient welfare or welfare without a market closure. Proposition 3 provides a formal statement comparing allocative efficiencies of different market structures.

PROPOSITION 3: *There always exists a $\Delta \in (0,1)$ such that the ex-ante welfare of a market design with a market closure, Proposition 1, is greater than that of a market design of* 24/7 *trading, Proposition 2.*

See Appendix .A.4 for the proof. It is worth noting that we prove Proposition 3 for the general model, where the correlation between inventory shocks can be non-zero, and correlations, volatility, and holding costs can differ between night and day. Therefore, the model suggests 24/7 trading is never optimal. There always exists some length of night, which might be very short, that helps coordinate liquidity and trade such that it more than makes up for the length of time where trade is restricted to not occur.

How long should the optimal closure be? The inability to trade is costly, but the threat of a longer nighttime and shorter trading day concentrates liquidity, which makes trade more efficient during the fraction of the day that the asset is traded. The answer, when there is an interior optimum, is characterized in Proposition 4. For tractability, we focus on the case where holding costs are the same across night and day, and the shocks to inventory are independent across traders, but the volatility of inventory shocks can be different between night and day.

PROPOSITION 4: *Assume $\gamma_d = \gamma_n$ and Brownian shocks are independent across traders. If there*

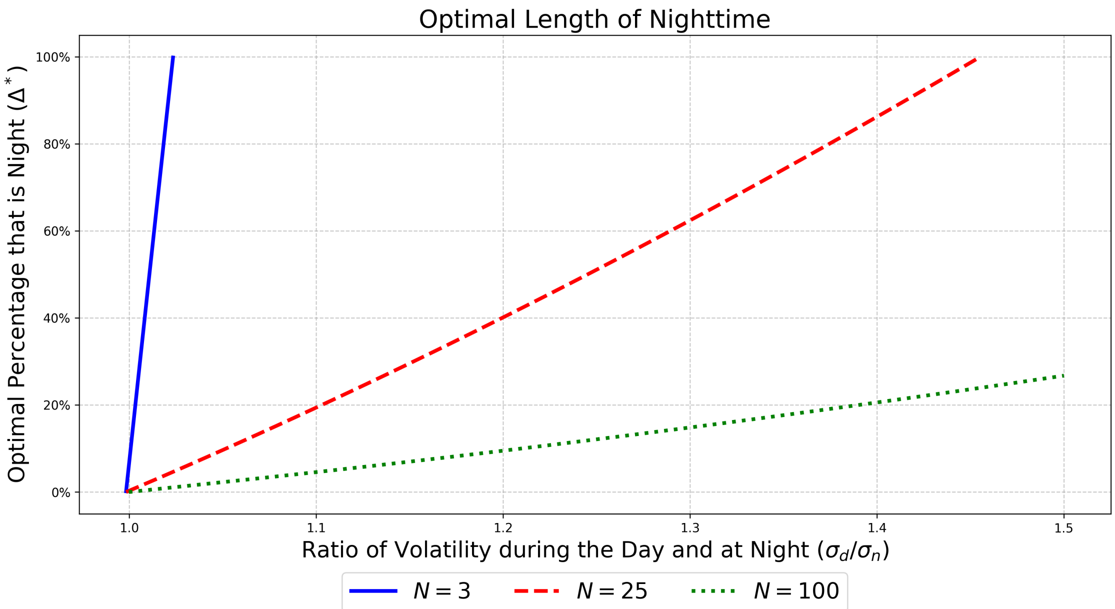


Figure 3. This figure plots the optimal fraction of a 24-hour period that the market should be closed as a function of the ratio of volatilities of inventory during the day and at night if an optimal length exists. The three different lines are for three different sizes of markets. $\lambda = 5\%$.

exists an optimal $\Delta^* \in (0, 1)$*, the unique solution is*

$$\Delta^* = \frac{(e^{\lambda}\left(\frac{\sigma_d}{\sigma_n}\right)^2 - 1)W_0\left(\frac{(N-2)e^{\frac{\lambda\left(\frac{\sigma_d}{\sigma_n}\right)^2-(1+\lambda)+e^{\lambda}(N-(1+\lambda)\left(\frac{\sigma_d}{\sigma_n}\right)^2)}{e^{\lambda}\left(\frac{\sigma_d}{\sigma_n}\right)^2-1}}}{e^{\lambda}\left(\frac{\sigma_d}{\sigma_n}\right)^2-1}\right) + \left(\frac{\sigma_d}{\sigma_n}\right)^2\left(e^{\lambda}(1+2\lambda)-\lambda\right) + 1 - e^{\lambda}N}{\lambda(e^{\lambda}\left(\frac{\sigma_d}{\sigma_n}\right)^2 - 1)}, \tag{17}$$

where W_0 *is the principal branch of the Lambert W function.*

Due to the interval $(0, 1)$ being non-compact, if the ratio of volatilities between day and night, $\frac{\sigma_d}{\sigma_n}$ becomes too far away from 1, either day or night is too desirable and there is no optimal length of closure in $(0, 1)$. The optimal length of closure is arbitrarily close to 0 if $\frac{\sigma_d}{\sigma_n} << 1$ and arbitrarily close to 1 if $\frac{\sigma_d}{\sigma_n} >> 1$. It is worth noting this result is not in contrast with Proposition 3, from which we know that there always exists some $\Delta > 0$ such that welfare is better than the 24/7 trading market structure.

When the ratio of volatilities is not too extreme, there exists an interior Δ^* that maximizes welfare. The optimal length of the trading day depends on three parameters: the number of traders in the market (N), the liquidation rate (λ), and the ratio of volatilities of individual inventory

(a) Welfare loss for a small market ($N = 10$)

(b) Welfare loss for a large market ($N = 100$)

Figure 4. These figures plot the loss in welfare relative to the first-best (efficient) welfare of different market designs. The left plot is for a small market ($N = 10$), while the right is for a larger market ($N = 100$). The blue solid line is the percentage loss for a market design with different lengths of market closures. The red dashed line is the percentage loss for the current market design of 6.5 trading hours a day. The dashed green line is the percentage loss for a market design that does not have a market closure and trades 24/7. Finally, $\lambda = 5\%$.

positions between the trading day and night ($\frac{\sigma_d}{\sigma_n}$). Figure 3 graphically plots equation (17), on the y-axis, for three different market sizes and for different ratios of the trading day to night volatilities, on the x-axis.

In small markets, where intraday liquidity is very thin, the benefit of the liquidity coordination of closure is substantial. Therefore, the optimal duration of the closure is decreasing in N. An extended closure is less costly due when inventory volatility is relatively lower overnight. Thus, the optimal duration of the closure is increasing in σ_d/σ_n. The incremental cost of higher overnight inventory volatility is lower in large, already liquid markets. This fact explains why the slopes of the curves in Figure 3 are greater in small markets. Finally, the optimal length of market closures does not exist when the ratio of volatilities is a little below one or well above one, depending on the size of the market. The optimum is arbitrarily close to one when inventory overnight has sufficiently low volatility, but not exactly one, as some non-zero trade is desirable. If overnight inventory volatility is higher than inventory volatility during the day, the optimal Δ is arbitrarily near 0, as a closure, however short, is desirable to coordinate liquidity.

A. The Cost of Imperfect Competition for Differing Closure Lengths

Proposition 3 shows that a market closure of some length is better than trading 24/7 trading, but it makes no remark statement regarding how much a market closure minimizes welfare loss due to the strategic behavior of traders relative to the first-best outcome. We also have not quantified how costly the predominant market hours for equities, trading for 6.5 hours a day, compares to shorter closures. Figure 4 provides some insights for these questions by plotting the welfare loss relative to W^e, equation (14), for $W^{24/7}$, equation (15), for a market closure of 17.5 hours, $W(17.5)$, and for differing lengths of a market closure, $W(\Delta)$.

First, as noted before, the 24/7 benchmark is only different from the efficient market by a factor of 2. This is why the small-dashed green line is at -100% for both panels. Starting with the smaller market in Figure 4(a), having a long closure like that which currently exists is much better than 24/7 trading, but decreasing the length of the closure to a shorter night would still potentially increase allocative efficiency. For a larger market, as in Figure 4(b), the current market closure length is more costly than 24/7 trading. In fact, a longer closure would be extremely costly. Additionally, a shorter market closure drastically further lowers the gap to the efficient benchmark. It is worth noting these results assume a constant volatility and holding costs across the day and night.

V. Intraday Patterns

Although not the primary focus of the paper, the model has the ability to explain some interesting intraday patterns that are observed in the data. We focus on two: intraday volume and returns.

A. Volume

A strong empirical pattern is the U-shaped (smirk) pattern of trading volume throughout the day (Chan et al., 1996, Jain and Joh, 1988). Despite the aggressiveness of each trader, $c(t)$, and price impact (liquidity) of the market, $b(t)$, being monotonic throughout the day, the model with market closures generates a U-shaped pattern in trading volume throughout the day. Define the

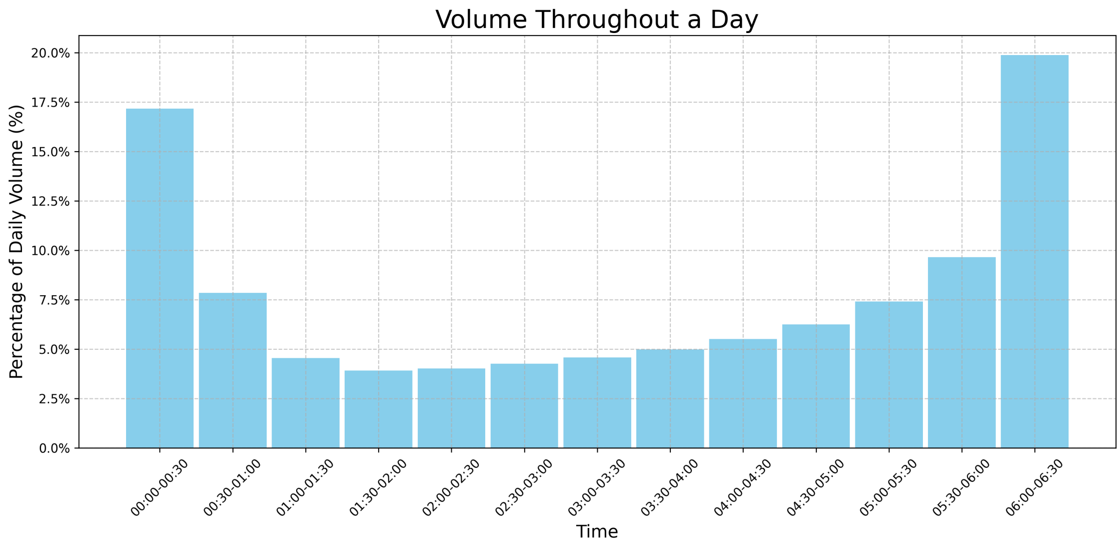


Figure 5. This figure is the percentage of the expected daily trading volume in each 30-minute bin when trading occurs for 6.5 hours a day. This example uses $N = 25$, $\lambda = 5\%$, $\sigma_d = \sigma_n$, and $\Delta = \frac{17.5}{24}$.

expected instantaneous trading volume as

$$\mathbb{E}\left[Volume_t\right] = \sum_{i=1}^{N} \mathbb{E}\left[|D_t^i|\right]$$
$$= \sqrt{\frac{2}{\pi}}|c(t)|\sqrt{N(N-1)\left(\Delta\sigma_n^2 e^{2\int_0^t c(s)ds} + \sigma_d^2 \int_0^t e^{2\int_s^t c(u)du} ds\right)}. \tag{18}$$

Due to the inability to trade overnight, the absolute gap between any trader's current and desired inventory position has an expected mean of $\sqrt{\frac{2(N-1)\Delta}{N\pi}}\sigma_n$ at the start of any given day. Therefore, although trade is not very aggressive in the morning in the sense that traders exchange a small percentage of the gap, due to the large average gap, they still trade a large quantity of the asset. In the middle of the day, they still are not very aggressive nor have a large excess inventory position. Finally, at the end of the day, they become very extremely aggressive and close the gap entirely, resulting in a large increase in trading volume.

Figure 5 demonstrates the above logic. Figure 5 plots the expected fraction of the total daily volume in each 30-minute trading bucket. To match the NYSE, we have 6.5 hours of trading a day. If trade volume was uniformly distributed throughout the day, you would expect about 7.7% of the daily volume in each bin. Yet, we see significantly more near the open and close. About 17% of the daily volume happens in the first 30 minutes, and about 20% happens in the last 30 minutes.

Some markets, such as foreign exchange (FX) markets or cryptocurrencies, already trade 24/7.

Yet, volume patterns in these markets are not flat throughout the day, as the equilibrium of Proposition 2 would imply. Empirically, we see spikes in volume in FX markets when either the London or New York stock exchanges are open, and especially during their overlap in opening hours. Likewise, for cryptocurrencies, we see volume rise when the London Stock Exchange opens and a further increase when the NYSE opens. Allowing volatility of inventory positions to be a deterministic function of time would likely match these patterns. As the instantaneous volatility increases, so would the instantaneous volume. Therefore, if market openings in other asset classes increase the volatility in a trader's inventory position, you could see these spikes in trading volume. Introducing another set of traders, who only trade between during hours of the day, such as when another market opens and closes, might also generate similar patterns in volume. The daily entry and departure of these traders could potentially coordinate trade sufficiently to generate the empirical patterns in markets that do trade 24/7. More generally, modeling the interdependence between exchanges and their hours is well beyond the scope of this paper; to be fully understood it would require the study of traders' dynamic strategic trade between correlated assets trading on different exchanges.

B. Returns

Recall the equilibrium price in the model is given by

$$p_t^* = v - \frac{2\gamma_d}{\lambda}(1 - e^{-r(1-\Delta-t)})\bar{Z}_t.$$

This implies, absent aggregate inventory shocks, open-to-close returns will be positive if $\bar{Z}_t > 0$ and negative if $\bar{Z}_t < 0$. Moreover, close-to-open returns will exhibit the opposite behavior.

A simple way to interpret these findings is the following. The price can, as is mentioned above, be viewed as the sum of fundamental value and a discount for aggregate holdings costs or risk. When aggregate inventory is positive, the price requires a discount in order to incentivize investors to hold the asset. This discount varies over time. As the day progresses, liquidity and price impact reduce, reducing effective inventory management costs and thus reducing the holding cost discount. At the end of the day, traders reach socially optimal inventories, and risk bearing capacity is maximized. Thus, an endogenous improvement in aggregate risk bearing capacity reduces the premium and increases prices throughout the day. When aggregate inventory is negative, improved risk bearing capacity leads to a decrease in prices over the trading day.

It's informative to interpret this result in terms of the results of Lu et al. (2023). The authors argue night-minus-day returns can be non-zero on average if risk bearing capacity varies systematically over the day, even if inventory imbalances do not reverse over the course of the day. In their model, risk-bearing capacity improves over the day as risk-averse "fast" traders with an informational advantage absorb informed order flow at the start of the day and later offload it to more risk-tolerant, but less informed, "slow" traders at the end of the day. In our model, risk bearing capacity improves towards the end of the day as homogeneous traders coordinate liquidity. Thus, if these traders, who can be viewed as liquidity providers, face net buying pressure throughout the day, our model will generate the standard finding that night-minus-day returns are positive (see e.g., Lu et al. (2023), Branch and Ma (2012), Kelly and Clark (2011)).

Although we feel the model can provide a simple means of explaining intraday average returns, it is worth noting that in the model the instantaneous volatility of prices decreases, which is inconsistent with commonly observed U-shaped intraday volatility. Although, we feel introducing two groups of traders to the model, one informed and another informed, would address this result. As the day progresses, information obtained overnight would be incorporated slowly into the price. If fundamental value were stochastic in the model, this could quite possibly result in a price at the end of the day that is more volatile as it reflects fundamental value more closely (see Hong and Wang (2000)). It is also worth noting such a model is fairly far afield from addressing our results regarding aggregate welfare. In the following section, we consider a model in which there is heterogeneous information in terms of private signals about an unobserved dividend, although there is not heterogeneity in information quality.

VI. Heterogeneous Information

In this section, we summarize an extension that allows for heterogeneous information regarding the level of dividend. The main results are analogous to those of previous sections, suggesting our results regarding the effect of a market closure on liquidity and allocative efficiency are robust to the consideration of information problems. The introduction of an information problem is done by adding two components to the model: a stochastic dividend, and private signals regarding the dividend. These components generate a learning problem, discussed below, on top of the inventory management problem discussed in detail in previous sections.

The dividend is now assumed to evolve according to $dv_t = \sigma_v dB_t^v$. Each trader continuously receives a signal S_t^i, where $dS_t^i = dv_t + \sigma_S^i dB_t^{iS}$. For simplicity, assume these Brownian shocks are all independent of each other, and of all shocks in the model. All other aspects of the model are the same as before.

We conjecture equilibrium demand schedules take the following form:

$$D^i(t, z^i, S^i, p) = a(t) + b(t)p + c(t)(z^{iI} + fz^{iD} + gS^i).$$

Based on these demand schedules, in equilibrium any investor will be able to observe $\bar{Z} + f\bar{S}$ directly from the price. Note there is no time dependence in f. This is a technical point, but an important one. If there were time dependence, investors' conditional expectations of the mean would no longer be a simple function of a few state variables, namely z^i, S^i and $\bar{Z} + f\bar{S}$. In particular, time dependence in f would effectively force beliefs to be a state variable of the problem. Any investor i's beliefs would depend on other investors' beliefs, which in turn depend on investor i's beliefs. This loop iterates, leading to an infinite regress of beliefs problem, which the literature has yet to understand how to resolve.

Given the above demand schedules, each investor solves a learning problem. Traders observe z^i, S^i and $\bar{Z} + f\bar{S}$, from which they infer the level of the dividend. In particular conditional beliefs of the dividend can be written as

$$E_t[v_t] = \bar{v} + C_1 z^{iI} + C_2 S^i + C_3(\bar{Z} + g\bar{S}),$$

for some constants C_1, C_2, C_3. Here, $z^i \equiv z^{iI} + z^{iD}$ is a decomposition of trader i's inventory into components due idiosyncratic shocks and due to previously traded components. Intuitively, only the idiosyncratic shocks component is informative regarding the level of the dividend, as their correlation with $\bar{Z}$ allow investors to better extract information about $\bar{S}$ from the price. C_1, C_2, and C_3 unsurprisingly depend on g, as the relative weight of the signal from the price on $\bar{Z}$ and $\bar{S}$ affects the learning problem. Conversely, g depends on C_1, C_2, C_3, as optimal demand schedules depend on beliefs. This fixed point problem leads to a straightforward non-linear equation for g.

Having set up the learning problem, we now characterize the equilibrium in Proposition 5. The results mirror the results in Proposition 1, accounting for the additional layer of complexity due to learning:

PROPOSITION 5: *There exists an unique symmetric, linear, and periodic equilibrium with the following properties for $t \in [0, 1-\Delta)$:*

1. *The optimal equilibrium demand schedule every trader plays is*

$$D^i(t, z^i, p) = \frac{\lambda\left((N-2)(2\gamma - \lambda C_1) + \lambda C_3\right)}{4(N-1)(\gamma - hf)\left(1 - e^{-\lambda(1-\Delta-t)}\right)} \times \left(\frac{\lambda f}{2(N-1)(\gamma - hf)(1 - e^{-\lambda(1-\Delta-t)})}(v-p) - z_t^{iI} - f z_t^{iD} - g S^i\right), \quad (19)$$

where f, g, h, C_1, C_3 are all constant, expressions for which are given in the Appendix.

2. *The equilibrium market clearing price is*

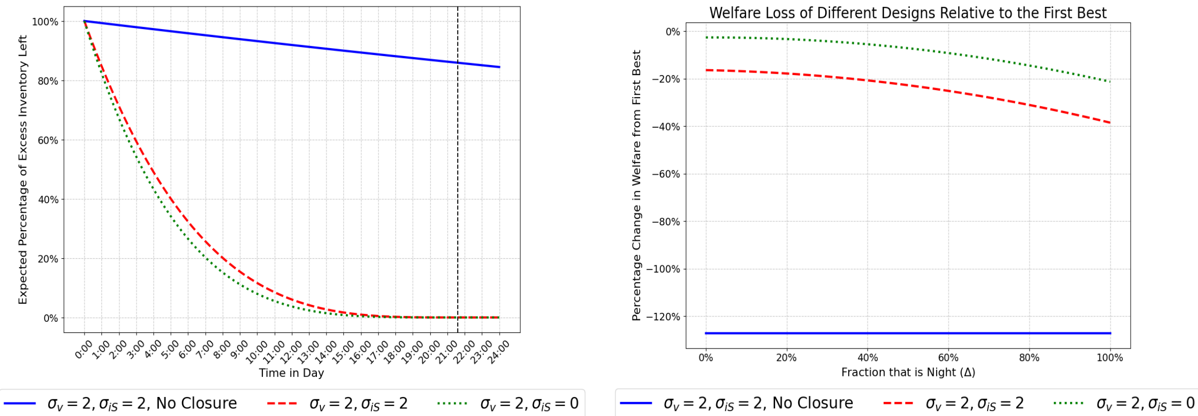
$$p_t^* = v - \frac{2(N-1)(\gamma - hf)}{\lambda f}(1 - e^{-\lambda(1-\Delta-t)})(\bar{Z}_t + g\bar{S}_t). \quad (20)$$

3. *Therefore, the equilibrium allocation per trader is*

$$D^i(t, z^i, p^*) = -\frac{\lambda\left((N-2)(2\gamma - \lambda C_1) + \lambda C_3\right)}{4(N-1)(\gamma - hf)\left(1 - e^{-\lambda(1-\Delta-t)}\right)}\left((z_t^{iI} + f z_t^{iD} - \bar{Z}_t) + g(S_t^i - \bar{S}_t)\right) \quad (21)$$

It is fairly straightforward to show that if the learning problem goes away, in the sense that $C_1 = C_2 = C_3 = 0$, these equilibrium demands and prices reduce to those of the main model in Proposition 1. More generally, as shown below in Figure 6, the main result of this paper still holds. As the trading day comes to an end, traders trade increasingly aggressively towards their desired allocations $-f z_t^{iD} + \bar{Z}_t + g(\bar{S}_t - S_t^i)$. As they do so, price impact decreases, further improving liquidity. This coordination of liquidity can also still be seen in prices, as improved risk bearing capacity is reflected in a premium for holding inventory that decreases throughout the trading day.

To gain some intuition for the the model of this section, we plot trading intensity and welfare in Figure 6. We consider the model of this section alongside a model in which σ_{iS} is set to 0, so that information asymmetry is eliminated, and alongside a model with information asymmetry but without market closure. In the left panel of Figure 6 we consider trading intensity by plotting $\exp(\int_0^t c(s)ds)$. As expressions in the Appendix illustrate, this quantity measures how much of the gap between a trader's inventory and desired inventory has closed in expectation between the start of the trading day and time t. For both models with closures, this number is 0 at the end of the day, consistent with the main model. Further, trading is significantly more aggressive throughout the day in the model with a closure, as liquidity is coordinated at the close. Perhaps unsurprisingly,



(a) Trading Intensity

(b) Welfare

Figure 6. These figures plot trading intensity and welfare for various different regimes. On the left, we plot $\exp(\int_0^t c(s)ds)$, the expression for the expected percentage of excess inventory left at point t in the day relative to the position at the start of the day, for a various market designs, assuming $\Delta = 0.1$ if there is a closure. The right plot plots ex-ante welfare at the start of the trading day. The blue solid line considers a market without closure and with information asymmetry. The red line considers a market with closure and information asymmetry. The green line considers a market with closure and without information asymmetry. We assume $N = 10$, $\lambda = 5\%$.

trading intensity with information asymmetry is slightly slower than without. It is worth noting that this slower trading is due primarily to heterogeneity, not simply uncertainty. If one plots the trading intensity corresponding to a model in which signals are perfectly correlated, it is nearly indistinguishable from the plot in which signals are public, not private.

In the right-hand panel, we see that market closure continues to have consequences for welfare. Without closure, welfare is worse when trading is continuous for the entire range of closure lengths. Thus, the primary mechanism of this paper continues to be present when there is heterogeneous information regarding asset values. Information asymmetry does tend to worsen welfare, which is unsurprising given the less aggressive trading and thus less liquid market illustrated by the results of the left-hand panel of Figure 6.

Although not the focus of this paper, it is worth discussing any implications the model might have for price efficiency. One can think of price efficiency as the magnitude of an investor's conditional variance of the dividend given their signals and the price, relative to the unconditional variance of the dividend, that is $\frac{\mathrm{Var}_t(v_t)}{\mathrm{Var}(v_t)}$. This value jumps down whenever trading opens, as investors infer information from the price, and increases whenever the market closes. Thus, market closure hinders price efficiency, simply because prices become stale overnight. Although worth pointing out, this is not a particularly surprising finding, as the information structure we consider is sufficiently simple and homogeneous to make the model tractable. For instance, our results suggest market closure helps coordinate liquidity near the close. If some traders were more informed than others, they might be more willing to trade, improving price efficiency. Can this mechanism outweigh the detrimental effect of closure on price efficiency that we document? More generally, the impact of market closure on the dynamic interaction between allocative efficiency, liquidity, and price efficiency under heterogeneously informed investors promises to yield very interesting research, which we leave to future study.

VII. Conclusion

Despite the rise of electronification in trading and execution rendering the historical reasons for the existence of market closures obsolete, this paper show that market closures play an important role in the allocative efficiency of market designs. Despite adding a constraint on when trade can occur, we show there always exists a market closure length such that trader welfare is higher than a market design where trade can occur 24/7. This result follows from strategic traders coordinate their aggressive trade in the lead-up to the market closure, which more than offsets the costs of the inability to trade during the closure itself. We further show that this result is robust to traders having heterogeneous information, which slows down trade, but does not eliminate our main channel.

While our model focuses on the effect of a market's opening hours on allocative efficiency, market closures may play an important role for many other reasons. Closing auction prices are used in the settlement of many derivative contracts, for margin requirements, performance of institutional investors, to price mutual fund shares, and the asset value for ETFs and stock indices. Our model does imply that the closing price is the fundamental value of the dividend. If trading were 24/7, there is no point in the day where this is true. Further, market closures have been used to make announcements without inducing excess short run volatility in a share price. Both the effect of market closure on the efficiency of closing prices and its interaction with endogenous disclosure decisions are important for policymakers and future research to consider.

Finally, the intersection of market closures and fragmentation is a potentially fruitful path we leave to future research. Traders can effectively trade 24/7 by using other international exchanges, such as the Tokyo or London Stock Exchanges. Yet, as noted in the main text, volume still clusters around the closes and openings of each exchange. Further, there are extended trading hours where trade can occur through electronic communication networks, or ECNs, yet these networks suffer from low liquidity and volatile prices. Circumventing closures on one exchange by routing trade through other means may limit the costs of limited trading hours for some securities. Moreover, if aggressive trade due to closure on one exchange leads to spillovers of aggressive trade onto other exchanges, some of the downsides corresponding to markets designed with 24/7 trading may be mitigated.

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Appendix

Appendix A. Derivation and Validation of the Value Function

Appendix A.1. When there is a Night ($\Delta > 0$)

We conjecture that the other $N-1$ traders submit demand schedules given by Equation 6. Trade is modeled by a uniform price double auction where the price is the solution to Equation 1. Therefore, the equilibrium price is

$$p_t^* = -\frac{a(t) + c(t)\bar{Z}_t}{b(t)}.$$

Given the equilibrium price, the demand schedule evaluated at the equilibrium price, rate allocated in equilibrium, is

$$D_t^i = c(t)\big(z_t^i - \bar{Z}_t\big).$$

Finally, conjecture the day value function takes the following linear-quadratic form[17]

$$J^d(t, z^i, \bar{Z}) = \alpha_0(t) + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3(t)(z^i)^2 + \alpha_4(t)\bar{Z}^2 + \alpha_5(t) z^i \bar{Z}.$$

Recall that traders rationally anticipate how their demand affects their trade price. Therefore, when trader i chooses demand d^i, they face the residual demand curve that, by market clearing, implies they face the price $\Phi(t, d^i, Z^{-i})$, defined in equation 7. Therefore, the Hamilton-Jacobi-Bellman equation is

$$\lambda J^d = \max_{d^i}\{J_t^d + \lambda z_t^i v - \Phi(t, d^i, Z^{-i})d^i - \gamma_d (z^i)^2 + J_{z^i}^d d^i + \frac{1}{2}J_{z^i z^i}^d \sigma_d^2 + \frac{1}{2}J_{\bar{Z}\bar{Z}}^d \sigma_{Z_d}^2 + J_{\bar{Z}z^i}^d \kappa_d\}.$$

Recall $\lambda = -\log(1-\mathcal{P})$, where $\mathcal{P}$ is the probability of a dividend payment in a given 24 hour period. First, we will solve for the equations that define the α functions and then will add in the optimality of demand constraints. Plugging the conjectured day value function into the HJB equation as well as the equilibrium price and demand schedule, we get

$$\begin{aligned}
&\lambda(\alpha_0(t) + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3(t)(z^i)^2 + \alpha_4(t)\bar{Z}^2 + \alpha_5(t) z^i \bar{Z}) \\
&\quad = \alpha_0'(t) + \alpha_3'(t)(z^i)^2 + \alpha_4'(t)\bar{Z}^2 + \alpha_5'(t) z^i \bar{Z} + z^i \lambda v + \frac{a(t) + c(t)\bar{Z}}{b(t)} c(t)(z^i - \bar{Z}) - \gamma_d (z^i)^2 \\
&\qquad + (\alpha_1 + 2\alpha_3(t) z^i + \alpha_5(t)\bar{Z}) c(t)\big(z^i - \bar{Z}\big) + \alpha_4(t)\sigma_{Z_d}^2 + \alpha_3(t)\sigma_d^2 + \alpha_5(t)\kappa_d.
\end{aligned}$$

[17] One can allow α_1, α_2 to be non-constant, although this generates a continuum of equilibria for which allocative efficiency, prices, and allocations are unchanged.

By matching coefficients, we get that

$$\begin{aligned}
\lambda\alpha_0(t) &= \alpha_0'(t) + \alpha_4(t)\sigma_{Z_d}^2 + \alpha_3(t)\sigma_d^2 + \alpha_5(t)\kappa_d \\
(\lambda - c(t))\alpha_1 &= \lambda v + \frac{a(t)c(t)}{b(t)} \\
\lambda\alpha_2 &= -\frac{a(t)c(t)}{b(t)} - \alpha_1 c(t) \\
(\lambda - 2c(t))\alpha_3(t) &= \alpha_3'(t) - \gamma_d \\
\lambda\alpha_4(t) &= \alpha_4'(t) - \frac{c(t)^2}{b(t)} - \alpha_5(t)c(t) \\
(\lambda - c(t))\alpha_5(t) &= \alpha_5'(t) + \frac{c(t)^2}{b(t)} - 2c(t)\alpha_3(t)
\end{aligned}$$

To get the optimality of demand equations, we take the first-order condition of the right side of the HJB equation with respect to d^i. This yields the equation

$$-\Phi - \Phi_{d^i} d^i + J_{z^i}^d = 0.$$

Plugging in the equilibrium expressions for Φ and d^i, we are left with the equations

$$\frac{a(t) + c(t)\bar{Z}}{b(t)} + \frac{1}{b(t)(N-1)} c(t)(z^i - \bar{Z}) + \alpha_1 + 2\alpha_3(t)z^i + \alpha_5(t)\bar{Z} = 0.$$

Matching coefficients in the above equation gives us three equations that must be satisfied for demand to be optimal,

$$\begin{aligned}
\frac{a(t)}{b(t)} + \alpha_1 &= 0, \\
\frac{c(t)}{b(t)} - \frac{c(t)}{b(t)(N-1)} + \alpha_5(t) &= 0, \\
\frac{c(t)}{b(t)(N-1)} + 2\alpha_3(t) &= 0.
\end{aligned}$$

Combining the optimality of demand equations with the α ODEs, they can be simplified to

$$\begin{aligned}
\lambda\alpha_0(t) &= \alpha_0'(t) + \alpha_4(t)\sigma_{Z_d}^2 + \alpha_3(t)\sigma_d^2 + \alpha_5(t)\kappa_d \\
\alpha_1 &= v \\
\alpha_2 &= 0 \\
(\lambda - 2c(t))\alpha_3(t) &= \alpha_3'(t) - \gamma_d \\
\lambda\alpha_4(t) &= \alpha_4'(t) + 2c(t)\alpha_3(t) \\
\lambda\alpha_5(t) &= \alpha_5'(t) - 4c(t)\alpha_3(t)
\end{aligned}$$

Note that the last equation can also be written as $(\lambda+\frac{2c(t)}{N-2})\alpha_5(t) = \alpha_5'(t)$. Also, from the optimality of demand equations, $\alpha_5(t) = 2(N-2)\alpha_3(t)$. Therefore, we have that

$$(\lambda - 2c(t))\alpha_3(t) = \alpha_3'(t) - \gamma_d$$
$$(\lambda + \frac{2c(t)}{N-2})2(N-2)\alpha_3(t) = 2(N-2)\alpha_3'(t).$$

Solving for $c(t)$ using the above two equations implies that

$$c(t) = \frac{\gamma_d(N-2)}{2(N-1)\alpha_3(t)}.$$

Given that $\alpha_3(t)$ and $\alpha_5(t)$ are constant multiples of each other, the boundary conditions will fail to hold if $c(t)$ is bounded. Thus, we assume $c(t)$ is unbounded below. We restrain $c(t)$ to be finite in $(0, 1-\Delta)$, but it can explode at the boundaries. This is equivalent to having $\alpha_3(t) = 0$ at one of the boundaries. If $\alpha_3(t) = 0$ at the left boundary, then the analysis below implies $c(t)$ will explode to infinity as $t \to 0^+$, which is not an allowable equilibrium. If $\alpha_3(t)$ is 0 at the right boundary, $c(t) \to -\infty$ as $t \to 1-\Delta^-$.

Now, solving the differential equations for the α's, we get

$$\alpha_0(t) = \frac{1}{\lambda^2}\frac{\gamma_d}{N-1}\left((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\right) - \left(A_4\sigma_{Z_d}^2 + A_3\sigma_d^2 + 2(N-2)A_3\kappa_d\right)te^{\lambda t} + A_0e^{\lambda t}$$
$$\alpha_1(t) = v$$
$$\alpha_2(t) = 0$$
$$\alpha_3(t) = A_3e^{\lambda t} - \frac{\gamma_d}{\lambda(N-1)}$$
$$\alpha_4(t) = A_4e^{\lambda t} + \frac{\gamma_d(N-2)}{\lambda(N-1)}$$
$$\alpha_5(t) = 2(N-2)A_3e^{\lambda t} - \frac{2\gamma_d(N-2)}{\lambda(N-1)}.$$

Since we know $\alpha_3(1-\Delta) = 0$, we can solve for A_3. This implies

$$A_3 = \frac{\gamma_d}{\lambda(N-1)}e^{\lambda(1-\Delta)}.$$

Now, we move on to the value function at night. Let $\hat{\bar{Z}}_t = E[\bar{Z}_s|t]$, the conditional expectation of the aggregate inventory given trader i's information at time t. We conjecture that the night value function takes the following linear-quadratic form

$$J^n(t, z^i, \hat{Z}) = \beta_0(t) + \beta_1 z_t^i + \beta_2\hat{Z}_t + \beta_3(t)(z_t^i)^2 + \beta_4(t)\hat{Z}_t^2 + \beta_5(t)z_t^i\hat{Z}_t.$$

Given the above-conjectured value function, the Hamilton-Jacobi-Bellman equation is

$$\lambda J^n = J^n_t - \gamma_n (z^i_t)^2 + \frac{1}{2} J^n_{z^i z^i} \sigma_n^2 + \frac{1}{2} J^n_{\hat{Z}\hat{Z}} \frac{(\rho^i_N)^2}{\sigma_n^2} + J^n_{\hat{Z} z^i} \kappa_n.$$

Plugging in the conjectured value function for a night, we get:

$$\lambda\big(\beta_0(t) + \beta_1 z^i_t + \beta_2 \hat{Z}_t + \beta_3(t)(z^i_t)^2 + \beta_4(t)\hat{Z}^2_t + \beta_5(t) z^i_t \hat{Z}_t\big)$$
$$= \beta_0'(t) + \beta_3'(t)(z^i_t)^2 + \beta_4'(t)\hat{Z}^2_t + \beta_5'(t) z^i_t \hat{Z}_t - \gamma_n (z^i_t)^2 + \beta_3(t)\sigma_n^2 + \beta_4(t)\frac{\kappa_n^2}{\sigma_n^2} + \beta_5(t)\kappa_n.$$

By matching coefficients, we get

$$\begin{aligned}
\lambda\beta_0(t) &= \beta_0'(t) + \beta_3(t)\sigma_n^2 + \beta_4(t)\frac{\kappa_n^2}{\sigma_n^2} + \beta_5(t)\kappa_n \\
\lambda\beta_1 &= 0 \\
\lambda\beta_2 &= 0 \\
\lambda\beta_3(t) &= \beta_3'(t) - \gamma_n \\
\lambda\beta_4(t) &= \beta_4'(t) \\
\lambda\beta_5(t) &= \beta_5'(t)
\end{aligned}$$

Solving the above ODEs yields the following equations

$$\begin{aligned}
\beta_0(t) &= B_0 e^{\lambda t} - \frac{\gamma_n \sigma_n^2}{\lambda^2} - \left(\kappa_n B_5 + \frac{\kappa_n^2}{\sigma_n^2} B_4 + \sigma_n^2 B_3\right) t e^{\lambda t} \\
\beta_1 &= 0 \\
\beta_2 &= 0 \\
\beta_3(t) &= -\frac{\gamma_n}{\lambda} + B_3 e^{\lambda t} \\
\beta_4(t) &= B_4 e^{\lambda t} \\
\beta_5(t) &= B_5 e^{\lambda t}
\end{aligned}$$

All that is left now is to use the boundary conditions to solve for the constants in the solutions for the α's and β's. Recall that the two boundary equations are $J^d(t = 1 - \Delta, z^i, \bar{Z}) = J^n(t = 1 - \Delta, z^i, \bar{Z})$ and $\lim_{t \to 1^-} J^n(t, z^i, \bar{Z}) = \lim_{t \to 1^-} \mathbb{E}\left[J^d(t = 1, z^i, \bar{Z}) \middle| \mathcal{I}_t\right]$. First consider the boundary condition at $t = 1$. To that end, we first compute expectation of J^d, given information at the end of the night. Each investor observes their inventory evolution throughout the night, which evolves according to

$$dz^i_t = \sigma_n dB^i_t.$$

Given the conditional covariance between Z_t and z_t^i is assumed to be κ_n, we can write

$$d\bar{Z}_t = \sigma_{Z_n}(\rho_N^i dB_t^i + \sqrt{1-(\rho_N^i)^2}dB_t^\epsilon)$$

where $\rho_N^i \sigma_n \sigma_{Z_n} = \kappa_n$. So, conditional on observing z_t^i, the conditional mean of $\bar{Z}_t$ is $\bar{Z}_{1-\Delta} + \frac{\kappa_n}{\sigma_n^2}(z_t^i - z_{1-\Delta}^i)$ and the conditional variance is $\sigma_{Z_n}^2(1-(\rho_N^i)^2)(t-(1-\Delta))$. Note that $\sigma_{Z_n}^2$ is the variance of $\bar{Z}$ and not Z. Therefore,

$$\lim_{t\to 1^-} \mathbb{E}\big[J^d(t=1, z^i, \hat{Z})\big|\mathcal{I}_t\big] = \alpha_0(0) + \alpha_4(0)\sigma_{Z_n}^2(1-(\rho_N^i)^2)\Delta + \alpha_1 z^i + \alpha_2 \hat{Z} + \alpha_3(0)(z^i)^2 + \alpha_4(0)\hat{Z}^2 + \alpha_5(0) z^i \hat{Z}.$$

Now consider the boundary condition at $t = 1-\Delta$. To this end, note in equilibrium if $t < 1-\Delta$,

$$d(z_t^i - \bar{Z}_t) = c(t)(z_t^i - \bar{Z}_t)dt + \sigma_d dB^i - \sigma_Z\left(\rho_N^i dB_t^i + \sqrt{1-(\rho_N^i)^2}dB_t\right),$$

where dB is a Brownian motion independent of dB^i. Thus,

$$z_t^i - \bar{Z}_t = e^{\int_0^t c(s)ds}(z_0^i - \bar{Z}_0) + \int_0^t e^{\int_u^t c(s)ds}\left(\rho_N^i dB_u^i + \sqrt{1-(\rho_N^i)^2}dB_u\right).$$

By the dominated convergence theorem, and based on the solution to c found above, $z_t^i - \bar{Z}_t^i \to 0$ as $t \to (1-\Delta)^-$. So, the state space contracts at $1-\Delta$, and we only need $\alpha_0(1-\Delta) = \beta_0(1-\Delta)$, $\alpha_3(1-\Delta)+\alpha_4(1-\Delta)+\alpha_5(1-\Delta) = \beta_3(1-\Delta)+\beta_4(1-\Delta)+\beta_5(1-\Delta)$. Note, $\alpha_3(1-\Delta) = \alpha_5(1-\Delta) = 0$.

Let's write out the boundary conditions in more detail. The boundary conditions for $\alpha_1, \alpha_2, \beta_1, \beta_2$ are matched trivially. The boundary conditions at $t = 1-\Delta$ reduce to

$$\frac{1}{\lambda^2}\frac{\gamma_d}{N-1}\Big((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\Big) - \Big(A_4\sigma_{Z_d}^2 + A_3\sigma_d^2 + 2(N-2)A_3\kappa_d\Big)(1-\Delta)e^{\lambda(1-\Delta)} + A_0 e^{\lambda(1-\Delta)}$$
$$= B_0 e^{\lambda(1-\Delta)} - \left(\kappa_n B_5 + \frac{\kappa_n^2}{\sigma_n^2}B_4 + \sigma_n^2 B_3\right)(1-\Delta)e^{\lambda(1-\Delta)} - \frac{\gamma_n \sigma_n^2}{\lambda^2}$$

$$A_4 e^{\lambda(1-\Delta)} + \frac{\gamma_d(N-2)}{\lambda(N-1)} = -\frac{\gamma_n}{\lambda} + B_3 e^{\lambda(1-\Delta)} + B_4 e^{\lambda(1-\Delta)} + B_5 e^{\lambda(1-\Delta)}.$$

Imposing periodicity of the value function, the boundary conditions at $t = 1$ reduce to

$$\frac{1}{\lambda^2}\frac{\gamma_d}{N-1}\Big((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\Big) + A_0 + \left(A_4 + \frac{\gamma_d(N-2)}{\lambda(N-1)}\right)\sigma_{Z_n}^2(1-(\rho_N^i)^2)\Delta$$
$$= B_0 e^{\lambda} - \left(\kappa_n B_5 + \frac{\kappa_n^2}{\sigma_n^2}B_4 + \sigma_n^2 B_3\right)e^{\lambda} - \frac{\gamma_n \sigma_n^2}{\lambda^2}$$

$$
\begin{aligned}
A_3 - \frac{\gamma_d}{\lambda(N-1)} &= -\frac{\gamma_n}{\lambda} + B_3 e^{\lambda} \\
A_4 + \frac{\gamma_d(N-2)}{\lambda(N-1)} &= B_4 e^{\lambda} \\
2(N-2)A_3 - \frac{2\gamma_d(N-2)}{\lambda(N-1)} &= B_5 e^{\lambda}
\end{aligned}
$$

Note, using $\alpha_3(1-\Delta) = 0$ yields A_3. Given A_3, solving for for A_4, B_3, B_4, and B_5 amounts to solving a linear system. The solutions are

$$
\begin{aligned}
A_4 &= \frac{(\gamma_n - \gamma_d)(N-1) + e^{-\lambda(1-\Delta)}(\gamma_d(2N-3) + e^{\lambda}(2\gamma_d + \gamma_n - N(\gamma_d + \gamma_n)))}{\lambda(e^{\lambda}-1)(N-1)}, \\
B_3 &= \frac{e^{-\lambda}(e^{-\lambda(1-\Delta)}\gamma_d - \gamma_d + \gamma_n - N\gamma_n)}{\lambda(N-1)}, \\
B_4 &= \frac{e^{-\lambda}((1-e^{-\lambda(1-\Delta)})\gamma_d(3 + e^{\lambda}(N-2) - 2N) - (e^{\lambda\Delta}-1)\gamma_n(N-1))}{(e^{\lambda}-1)\lambda(N-1)}, \\
B_5 &= -\frac{2\gamma_d(N-2)e^{-\lambda}(1-e^{-\lambda(1-\Delta)})}{\lambda(N-1)}.
\end{aligned}
$$

Lastly, A_0, B_0 are pinned down by the end of day and end of night equations involving those two variables.

Finally, using the optimality of demand equations and our solution for α_3 and α_5, we get that

$$
a(t) = \frac{v\lambda^2(N-2)}{4\gamma_d\big(1-e^{-\lambda(1-\Delta-t)}\big)^2}, \tag{22}
$$

$$
b(t) = -\frac{\lambda^2(N-2)}{4\gamma_d\big(1-e^{-\lambda(1-\Delta-t)}\big)^2}, \tag{23}
$$

$$
c(t) = -\frac{\lambda(N-2)}{2\big(1-e^{-\lambda(1-\Delta-t)}\big)}. \tag{24}
$$

Therefore, the equilibrium price, demand schedule submitted, and allocation are

$$
p_t^* = v - \frac{2\gamma_d}{\lambda}\big(1-e^{-\lambda(1-\Delta-t)}\big)\bar{Z}_t, \tag{25}
$$

$$
D_t^i(p) = \frac{\lambda(N-2)}{2\big(1-e^{-\lambda(1-\Delta-t)}\big)}\left(\frac{\lambda(v-p)}{2\gamma_d(1-e^{-\lambda(1-\Delta-t)})} - z_t^i\right), \tag{26}
$$

$$
D_t^i(p_t^*) = -\frac{\lambda(N-2)}{2\big(1-e^{-\lambda(1-\Delta-t)}\big)}\left(z_t^i - \bar{Z}_t\right). \tag{27}
$$

While the solutions for some of the α functions can be unwieldy, in the case where $\gamma_d = \gamma_n$ and $\sigma_d = \sigma_n$, the formulas are much nicer. Under those two assumptions, the day value function is

$$
J^d(t, z^i, \bar{Z}) = \alpha_0(t) + vz^i - \frac{\gamma_d}{\lambda(N-1)}\big(1-e^{-\lambda(1-\Delta-t)}\big)(z^i)^2
$$

$$+\left(\frac{\gamma_d(N-2)}{\lambda(N-1)}\big(1-e^{-\lambda(1-\Delta-t)}\big)-\frac{\gamma_d}{\lambda}e^{-\lambda(1-\Delta-t)}\right)\bar{Z}^2-\frac{2\gamma_d(N-2)}{\lambda(N-1)}\big(1-e^{-\lambda(1-\Delta-t)}\big)z^i\bar{Z}$$

This simplifies slightly to

$$J^d(t,z^i,\bar{Z})=\alpha_0(t)+vz^i-\frac{\gamma_d(1-e^{-\lambda(1-\Delta-t)})}{\lambda}(z^i)^2-\frac{\gamma_d}{\lambda}e^{-\lambda(1-\Delta-t)}\bar{Z}^2+\frac{\gamma_d(N-2)(1-e^{-\lambda(1-\Delta-t)})}{\lambda(N-1)}\big(z^i-\bar{Z}\big)^2.$$

For ex-ante welfare comparisons, we need the value function at the start of the day, $t=0$. I will also assume that the correlation between individual shocks is zero. Using this, the ex-ante expected value function is

$$\begin{aligned}\mathbb{E}\Big[J^d(0,z^i,\bar{Z})\Big] &= \frac{-2\gamma_d\sigma_d^2}{\lambda^2 N}+\frac{\gamma_d\sigma_d^2\Big(e^{-\lambda(1-\Delta)}(\lambda\Delta+e^{\lambda}(2+\lambda(1-\Delta)-N))+(N-2)(1+\lambda\Delta)\Big)}{\lambda^2N(e^{\lambda}-1)}\\ &\quad -\frac{\gamma_d(1-e^{-\lambda(1-\Delta)})}{\lambda}\sigma_d^2-\frac{\gamma_d}{\lambda}e^{-\lambda(1-\Delta)}\frac{\sigma_d^2}{N}+\frac{\gamma_d(N-2)(1-e^{-\lambda(1-\Delta)})}{\lambda}\frac{\sigma_d^2}{N}.\end{aligned}$$

This simplifies slightly to

$$\mathbb{E}\Big[J^d(0,z^i,\bar{Z})\Big]=\frac{\gamma_d\sigma_d^2}{\lambda N}\left[\frac{-2(1+\lambda)}{\lambda}+\frac{\Big(e^{-\lambda(1-\Delta)}(\lambda\Delta+e^{\lambda}(2+\lambda(1-\Delta)-N))+(N-2)(1+\lambda\Delta)\Big)}{\lambda(e^{\lambda}-1)}+e^{-\lambda(1-\Delta)}\right].$$

Appendix A.2. Benchmark: When there is Not a Night ($\Delta = 0$)

This section solves for the equilibrium when there is continuous trading in the model at all times, $\Delta = 0$. As we will show, this equilibrium is a special case of Du and Zhu (2017), where the time between trades goes to zero, and there is no adverse selection, and Antill and Duffie (2020), when the rate of occurrence of a size-discovery session occurs is set to zero. Unlike these above two papers, we will conjecture a time-varying strategy that is periodic across days, as in Equation 6. We will see, though, that the unique equilibrium solution is constant throughout the day and that of prior literature that conjures stationary, symmetric, and linear strategies.

We conjecture that the other $N-1$ traders submit demand schedules given by Equation 6. Trade is modeled by a uniform price double auction where the price is the solution to Equation 1. Therefore, the equilibrium price is

$$p_t^*=-\frac{a(t)+c(t)\bar{Z}_t}{b(t)}.$$

Given the equilibrium price, the demand schedule evaluated at the equilibrium price is

$$D_t^i=c(t)\big(z_t^i-\bar{Z}_t\big).$$

Finally, conjecture the day value function takes the following linear-quadratic form[18]

$$J^d(t, z^i, \bar{Z}) = \alpha_0(t) + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3(t)(z^i)^2 + \alpha_4(t)\bar{Z}^2 + \alpha_5(t) z^i \bar{Z}.$$

Recall that traders rationally anticipate how their demand affects their trade price. Therefore, when trader i chooses demand d^i, they face the residual demand curve that, by market clearing, implies they face the price $\Phi(t, d^i, Z^{-i})$, defined in equation 7. Therefore, the Hamilton-Jacobi-Bellman equation is

$$\lambda J^d = \max_{d^i}\{J_t^d + \lambda z_t^i v - \Phi(t, d^i, Z^{-i})d^i - \gamma_d (z^i)^2 + J_{z^i}^d d^i + \frac{1}{2} J_{z^i z^i}^d \sigma_d^2 + \frac{1}{2} J_{\bar{Z}\bar{Z}}^d \sigma_{Z_d}^2 + J_{\bar{Z} z^i}^d \kappa_d\}.$$

First, we will solve for the equations that define the α functions and then will add in the optimality of demand constraints. Plugging the conjectured day value function into the HJB equation as well as the equilibrium price and demand schedule, we get

$$\begin{aligned}
&\lambda(\alpha_0(t) + \alpha_1 z^i + \alpha_2 \bar{Z} + \alpha_3(t)(z^i)^2 + \alpha_4(t)\bar{Z}^2 + \alpha_5(t) z^i \bar{Z}) \\
&\quad = \alpha_0'(t) + \alpha_3'(t)(z^i)^2 + \alpha_4'(t)\bar{Z}^2 + \alpha_5'(t) z^i \bar{Z} + \lambda z^i v + \frac{a(t) + c(t)\bar{Z}}{b(t)} c(t)(z^i - \bar{Z}) - \gamma_d (z^i)^2 \\
&\qquad + (\alpha_1 + 2\alpha_3(t) z^i + \alpha_5(t)\bar{Z}) c(t)\big(z^i - \bar{Z}\big) + \alpha_4(t)\sigma_{Z_d}^2 + \alpha_3(t)\sigma_d^2 + \alpha_5(t)\kappa_d.
\end{aligned}$$

By matching coefficients, we get that

$$\begin{aligned}
\lambda \alpha_0(t) &= \alpha_0'(t) + \alpha_4(t)\sigma_{Z_d}^2 + \alpha_3(t)\sigma_d^2 + \alpha_5(t)\kappa_d \\
(\lambda - c(t))\alpha_1 &= \lambda v + \frac{a(t)c(t)}{b(t)} \\
\lambda \alpha_2 &= -\frac{a(t)c(t)}{b(t)} - \alpha_1 c(t) \\
(\lambda - 2c(t))\alpha_3(t) &= \alpha_3'(t) - \gamma_d \\
\lambda \alpha_4(t) &= \alpha_4'(t) - \frac{c(t)^2}{b(t)} - \alpha_5(t)c(t) \\
(\lambda - c(t))\alpha_5(t) &= \alpha_5'(t) + \frac{c(t)^2}{b(t)} - 2c(t)\alpha_3(t)
\end{aligned}$$

To get the optimality of demand equations, we take the first-order condition of the right side of the HJB equation with respect to d^i. This yields the equation

$$-\Phi - \Phi_{d^i} d^i + J_{z^i}^d = 0.$$

[18]One can allow α_1, α_2 to be non-constant, but they would be constant in equilibrium regardless. We conjecture that they are constant to stay consistent with Proposition 1.

Plugging in the equilibrium expressions for Φ and d^i, we are left with the equations

$$\frac{a(t) + c(t)\bar{Z}}{b(t)} + \frac{1}{b(t)(N-1)}c(t)(z^i - \bar{Z}) + \alpha_1 + 2\alpha_3(t)z^i + \alpha_5(t)\bar{Z} = 0.$$

Matching coefficients in the above equation gives us three equations that must be satisfied for demand to be optimal,

$$\begin{aligned} \frac{a(t)}{b(t)} + \alpha_1 &= 0, \\ \frac{c(t)}{b(t)} - \frac{c(t)}{b(t)(N-1)} + \alpha_5(t) &= 0, \\ \frac{c(t)}{b(t)(N-1)} + 2\alpha_3(t) &= 0. \end{aligned}$$

Combining the optimality of demand equations with the α ODEs, they can be simplified to

$$\begin{aligned} \lambda\alpha_0(t) &= \alpha_0'(t) + \alpha_4(t)\sigma_{Z_d}^2 + \alpha_3(t)\sigma_d^2 + \alpha_5(t)\kappa_d \\ \alpha_1 &= v \\ \alpha_2 &= 0 \\ (\lambda - 2c(t))\alpha_3(t) &= \alpha_3'(t) - \gamma_d \\ \lambda\alpha_4(t) &= \alpha_4'(t) + 2c(t)\alpha_3(t) \\ \lambda\alpha_5(t) &= \alpha_5'(t) - 4c(t)\alpha_3(t) \end{aligned}$$

Note that the last equation can also be written as $(\lambda + \frac{2c(t)}{N-2})\alpha_5(t) = \alpha_5'(t)$. Also, from the optimality of demand equations, $\alpha_5(t) = 2(N-2)\alpha_3(t)$. Therefore, we have that

$$\begin{aligned} (\lambda - 2c(t))\alpha_3(t) &= \alpha_3'(t) - \gamma_d \\ (\lambda + \frac{2c(t)}{N-2})2(N-2)\alpha_3(t) &= 2(N-2)\alpha_3'(t). \end{aligned}$$

Solving for $c(t)$ using the above two equations implies that

$$c(t) = \frac{\gamma_d(N-2)}{2(N-1)\alpha_3(t)}.$$

First, let us consider the case that $c(t)$ is unbounded at one of the endpoints of $(0,1)$. As in the market closure case, if $c(t)$ diverges at the end of the day, then $\alpha_3(1) = 0$. But, the boundary conditions are that $\alpha_i(0) = \alpha_i(1)$, for $i \in \{0, \ldots, 5\}$. Therefore, it must be the case that $\alpha_3(0) = 0$ too. But, as seen below, there will not exist an A_3 that would let $\alpha_3(0) = \alpha_3(1) = 0$. Therefore, a diverging $c(t)$ can not be an equilibrium. We, therefore, solve for the unique equilibrium, which is when $c(t)$ is finite.

Now, solving the differential equations for the α's, we get

$$\alpha_0(t) = \frac{1}{\lambda^2}\frac{\gamma_d}{N-1}\left((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\right) - \left(A_4\sigma_{Z_d}^2 + A_3\sigma_d^2 + 2(N-2)A_3\kappa_d\right)te^{\lambda t} + A_0 e^{\lambda t}$$
$$\alpha_1(t) = v$$
$$\alpha_2(t) = 0$$
$$\alpha_3(t) = A_3 e^{\lambda t} - \frac{\gamma_d}{\lambda(N-1)}$$
$$\alpha_4(t) = A_4 e^{\lambda t} + \frac{\gamma_d(N-2)}{\lambda(N-1)}$$
$$\alpha_5(t) = 2(N-2)A_3 e^{\lambda t} - \frac{2\gamma_d(N-2)}{\lambda(N-1)}.$$

Recall that the boundary equations are $J^d(t=1, z^i, \bar{Z}) = J^d(t=0, z^i, \bar{Z})$. This gives us

$$\frac{1}{\lambda^2}\frac{\gamma_d}{N-1}\left((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\right) - \left(A_4\sigma_{Z_d}^2 + A_3\sigma_d^2 + 2(N-2)A_3\kappa_d\right)e^{\lambda} + A_0 e^{\lambda}$$
$$= \frac{1}{\lambda^2}\frac{\gamma_d}{N-1}\left((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\right) + A_0,$$

$$A_3 e^{\lambda} - \frac{\gamma_d}{\lambda(N-1)} = A_3 - \frac{\gamma_d}{\lambda(N-1)}$$
$$A_4 e^{\lambda} + \frac{\gamma_d(N-2)}{\lambda(N-1)} = A_4 + \frac{\gamma_d(N-2)}{\lambda(N-1)},$$

The solutions for A_0, A_3, and A_4 are trivially

$$A_0 = 0,$$
$$A_3 = 0,$$
$$A_4 = 0.$$

Finally, using the optimality of demand equations and our solution for α_3 and α_5, we get that

$$a(t) = \frac{v\lambda^2(N-2)}{4\gamma_d}, \tag{28}$$
$$b(t) = -\frac{\lambda^2(N-2)}{4\gamma_d}, \tag{29}$$
$$c(t) = -\frac{\lambda(N-2)}{2}. \tag{30}$$

Therefore, the equilibrium price, allocation, and value function are

$$p_t^* = v - \frac{2\gamma_d}{\lambda}\bar{Z}_t,$$

$$D^i(t, z^i, p^*) = -\frac{\lambda(N-2)}{2}(z_t^i - \bar{Z}_t),$$

$$J(z^i, Z) = \frac{\gamma_d}{\lambda^2(N-1)}\Big((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\Big) + vz^i - \frac{\gamma_d}{\lambda(N-1)}(z^i)^2 + \frac{\gamma_d(N-2)}{\lambda(N-1)}\bar{Z}^2 - \frac{2\gamma_d(N-2)}{\lambda(N-1)}z^i\bar{Z}.$$

The value function simplifies slightly to

$$J(z^i, Z) = \frac{\gamma_d}{\lambda^2(N-1)}\Big((N-2)\sigma_{Z_d}^2 - \sigma_d^2 - 2(N-2)\kappa_d\Big) + vz^i - \frac{\gamma_d}{\lambda}(z^i)^2 + \frac{\gamma_d(N-2)}{\lambda(N-1)}\big(z^i - \bar{Z}\big)^2.$$

For ex-ante welfare comparisons, we need the expected value function before trade. I will also assume that the correlation between individual shocks is zero.

$$\mathbb{E}\Big[J(z^i, \bar{Z})\Big] = \frac{-2\gamma_d\sigma_d^2}{\lambda^2 N} - \frac{\gamma_d}{\lambda}\sigma_d^2 + \frac{\gamma_d(N-2)}{\lambda N}\sigma_d^2.$$

This simplifies to

$$\mathbb{E}\Big[J(z^i, \bar{Z})\Big] = \frac{-2\gamma_d\sigma_d^2(1+\lambda)}{\lambda^2 N}.$$

Appendix A.3. The First-Best Allocations

The first best allocation and maximal welfare achievable by any market design would be when a social planner continuously reallocated the aggregate inventory such that each trader always held the perfect risk-sharing amount. This would also be the welfare if the market was perfectly competitive and trade occurred 24/7. The value function for a trader who constantly holds the perfect risk-sharing amount is

$$J^e(\bar{Z}) = \mathbb{E}\Big[v\bar{Z}_{\mathcal{T}} - \int_0^{\mathcal{T}} \gamma \bar{Z}_t^2 dt\Big].$$

Taking the expectation first, the above equation can be rewritten as

$$J^e(\bar{Z}) = \lambda v\bar{Z}\int_0^{\infty} e^{-\lambda t}dt - \lambda\gamma\int_0^{\infty} e^{-\lambda t}\int_0^t \Big(\bar{Z}^2 + \frac{\sigma^2}{N}s\Big)ds dt.$$

Evaluating the integrals gives us that the value function is

$$J^e(\bar{Z}) = v\bar{Z} - \frac{\gamma}{\lambda}\Big(\bar{Z}^2 + \frac{\sigma^2}{\lambda N}\Big).$$

Taking the ex-ante expectation, given that $\bar{Z} \sim N(0, \frac{\sigma^2}{N})$, and summing across all traders, gives us that the maximal welfare achievable in this model is

$$\sum_{i=1}^{N}\mathbb{E}\big[J^e(\bar{Z})\big] = \frac{-\gamma\sigma^2(1+\lambda)}{\lambda^2}.$$

Appendix A.4. When is Welfare Improved by Having a Nighttime?

We define welfare as the sum of the ex-ante expectation of all trader's value functions. When there is a night, welfare can be expressed as

$$\sum_{i=1}^{N} \mathbb{E}\left[J^d(0, z^i, \bar{Z})\right] = \sum_{i=1}^{N} \left(\alpha_0(0) + \alpha_3(0)\sigma_d^2 + \alpha_4(0)\sigma_{Z_d}^2 + \alpha_5(0)\kappa_d\right).$$

Note that $\sigma_{Z_d}^2 = \kappa_d$. When there is no market closure and trade occurs 24/7, welfare can be expressed as

$$\sum_{i=1}^{N} \mathbb{E}\left[J(z^i, \bar{Z})\right] = \frac{-\gamma_d N\big(\sigma_d^2 + (N-2)\kappa_d\big)}{\lambda^2(N-1)} - \frac{\gamma_d N}{\lambda}\sigma_d^2 + \frac{\gamma_d N(N-2)}{\lambda(N-1)}\big(\sigma_d^2 - \kappa_d\big).$$

Therefore, the posed question of when is having a market closure beneficial is equivalent to when is $\sum_{i=1}^{N} \mathbb{E}[J^d(0, z^i, \bar{Z})] > \sum_{i=1}^{N} \mathbb{E}[J(z^i, \bar{Z})]$. This will always be true for at least some arbitrarily small Δ. To see this, take the limit as $\Delta \to 0^+$, then

$$\lim_{\Delta \to 0^+} \sum_{i=1}^{N} \mathbb{E}[J^d(0, z^i, \bar{Z})] - \sum_{i=1}^{N} \mathbb{E}[J(z^i, \bar{Z})] = \frac{\gamma_d N(2 - e^{-\lambda})(\sigma_d^2 - \sigma_{Z_d}^2)}{(N-1)\lambda(e^{\lambda} - 1)} \geq 0.$$

It is a strict inequality as long as the shocks across traders are not perfectly positively correlated.

Appendix B. What is the Optimal Length of a Trading Day?

To find the optimal length of the trading day, we maximize the value function in Δ. We assume that inventory shocks are independent across traders and the holding costs are constant across night and day, but allow for the volatility of the inventory shocks to differ across night and day. In Appendix .A.4, we showed some $\Delta > 0$ is always better than $\Delta = 0$. Therefore, we solve for

$$\Delta^* = \arg\max_{\Delta \in (0,1)} \sum_{i=1}^{N} \mathbb{E}\left[J^d(0, z^i, \bar{Z})\right].$$

The first-order condition with respect to Δ, and simplifying out any positive constants, is

$$(N-2)\sigma_n^2 - e^{-\lambda(1-\Delta)}(\lambda\sigma_d^2 - \sigma_n^2(1 + \lambda\Delta) + e^{\lambda}((-1 + (\Delta - 2)\lambda)\sigma_d^2 + N\sigma_n^2)) = 0.$$

Define W_0 to be the principal branch of the Lambert W function. Then, the optimal length of night is

$$\Delta^* = \frac{(e^\lambda \left(\frac{\sigma_d}{\sigma_n}\right)^2 - 1) W_0 \left(\frac{(N-2) e^{\frac{\lambda\left(\frac{\sigma_d}{\sigma_n}\right)^2 - (1+\lambda) + e^\lambda (N - (1+\lambda)\left(\frac{\sigma_d}{\sigma_n}\right)^2)}{e^\lambda \left(\frac{\sigma_d}{\sigma_n}\right)^2 - 1}}}{e^\lambda \left(\frac{\sigma_d}{\sigma_n}\right)^2 - 1} \right) + \left(\frac{\sigma_d}{\sigma_n}\right)^2 \left(e^\lambda (1+2\lambda) - \lambda\right) + 1 - e^\lambda N}{\lambda(e^\lambda \left(\frac{\sigma_d}{\sigma_n}\right)^2 - 1)}.$$

Appendix C. Heterogeneous information

Throughout, we will assume inventory costs and volatilities of shocks to signals and fundamentals are the same overnight as during the trading day. Simple extensions of the model can accommodate changes in these parameters between night and day.

For this problem, conjecture demand functions of the following form:

$$D_t^i(t, z^{iI}, z^{iD}, S^i) = a(t) + b(t)p + c(t)(z^{iI} + f z^{iD} + g S^i).$$

It is straightforward to see that equilibrium allocations will be

$$D_t^i = c(t)((z_t^{iI} + f z_t^{iD} - \bar{Z}_t) + g(S_t^i - \bar{S}_t)),$$

the residual demand curve is

$$\Phi(t, d^i, z^{-iI}, z^{-iD}, S^{-i}) = -\frac{1}{b(t)(N-1)} \left(d^i + (N-1)a(t) + c(t) \sum_{j \neq i} z^{jI} + c(t) f \sum_{j \neq i} z^{jD} + c(t) g \sum_{j \neq i} S^i \right),$$

and the equilibrium price is

$$-\frac{a(t) + c(t)(\bar{Z}_t + g\bar{S}_t)}{b(t)}.$$

Recall the liquidating dividend has arrival rate λ. Its value evolves according to $dv_t = \sigma_v dB^v$, and agents receive idiosyncratic signals $dS^i = \sigma_v dB^v + \sigma_{iS} dB^{iS}$. Then, we need to solve a learning problem. Traders observe their own signal S^i and observe $\bar{Z} + g\bar{S}$ from the price.

We begin by solving the learning problem. During the day, we can consider the following state variables for the learning problem

$$\theta = \begin{bmatrix} v \\ \bar{Z} \\ \bar{S} \end{bmatrix}$$

and the observation equation is

$$\xi = \begin{bmatrix} \bar{Z} + g\bar{S} \\ z^{iI} \\ S^i \end{bmatrix}$$

Then,

$$d\theta = \begin{pmatrix} \sigma_v & 0 & 0 & 0 & 0 \\ 0 & \sigma_Z \rho_N^i & \sigma_Z \sqrt{1-(\rho_N^i)^2} & 0 & 0 \\ \sigma_v & 0 & 0 & \sigma_S \rho_{NS}^i & \sigma_S \sqrt{1-(\rho_{NS}^i)^2} \end{pmatrix} \begin{pmatrix} dB^v \\ dB^i \\ dB \\ dB^{iS} \\ dB^S \end{pmatrix}$$

and

$$d\xi = \begin{pmatrix} g\sigma_v & \sigma_Z \rho_N^i & \sigma_Z \sqrt{1-(\rho_N^i)^2} & g\sigma_S \rho_{NS}^i & g\sigma_S \sqrt{1-(\rho_{NS}^i)^2} \\ 0 & \sigma & 0 & 0 & 0 \\ \sigma_v & 0 & 0 & \sigma_{iS} & 0 \end{pmatrix} \begin{pmatrix} dB^v \\ dB^i \\ dB \\ dB^{iS} \\ dB^S \end{pmatrix}$$

Above, the Brownian motions are defined in such a way that they are independent. Rational learning implies

$$dE_t(\theta_t) = \begin{pmatrix} g(\sigma_v)^2 & & \sigma_v^2 \\ \sigma_Z^2 & \sigma\sigma_Z\rho_N^i & \\ g(\sigma_v^2+\sigma_S^2) & & \sigma_v^2+\sigma_S\sigma_{iS}\rho_{NS}^i \end{pmatrix} \begin{pmatrix} g^2\sigma_v^2+\sigma_Z^2+g^2\sigma_S^2 & \sigma\sigma_Z\rho_N^i & g(\sigma_v^2+\sigma_S\sigma_{iS}\rho_{NS}^i) \\ \sigma\sigma_Z\rho_N^i & \sigma^2 & \\ g(\sigma_v^2+\sigma_S\sigma_{iS}\rho_{NS}^i) & & \sigma_v^2+\sigma_{iS}^2 \end{pmatrix}^{-1} d\xi$$

Then, integrating, $E_t[v] = \bar{v} + C_1 z^{iI} + C_2 S^i + C_3 G$, for some constants which are functions of g. Below, we show that $g = \frac{\lambda C_2}{\lambda C_1 - 2\gamma}$, so that this simple equation determines g and thus C_1, C_2, C_3. In the numerical examples considered, g had a unique solution.

Now let's move to the solution of trader i's optimization probelm. Call the value function $J^d(t, z^{iI} \equiv \int_0^t \sigma dB^i, z^{iD} \equiv \int_0^t D^i dt, S^i, G \equiv \bar{Z} + e\bar{S})$, and conjecture

$$\begin{aligned} J^d(t, z^{iI}, z^{iD}, S^i, G) = {} & \alpha_0(t) + \alpha_1(z^{iI} + z^{iD}) + \alpha_2(t)(z^{iI})^2 + \alpha_3(t)(z^{iD})^2 + \alpha_4(t)(S^i)^2 + \alpha_5(t)G^2 + \\ & + \alpha_6(t) z^{iI} z^{iD} + \alpha_7(t) z^{iI} S^i + \alpha_8(t) z^{iI} G \\ & + \alpha_9(t) z^{iD} S^i + \alpha_{10}(t) z^{iD} G + \alpha_{11}(t) G S^i. \end{aligned}$$

The HJB equation is

$$\begin{aligned} \lambda J = {} & J_t^d + \lambda(z^{iI} + z^{iD})(\bar{v} + C_1 z^{iI} + C_2 S^i + C_3 G) + \Phi_{d^i} d^2 - \gamma (z^{iI} + z^{iD})^2 \\ & + \frac{1}{2} J^d_{z^{iI} z^{iI}} \sigma^2 + \frac{1}{2} J^d_{S^i S^i}(\sigma_v^2 + \sigma_{iS}^2) + \frac{1}{2} J^d_{GG}(\sigma_Z^2 + g^2\sigma_v^2 + g^2\sigma_S^2) \\ & + J^d_{z^{iI} G} \rho_N^i \sigma \sigma_Z + J^d_{S^i G}\, g(\sigma_v^2 + \sigma_{iS}\sigma_S \rho_{NS}^i). \end{aligned}$$

By algebra similar to above,

$$
\begin{aligned}
\lambda\alpha_0 &= \alpha_0'(t) + \alpha_2(t)\sigma^2 + \alpha_4(t)(\sigma_v^2 + \sigma_{iS}^2) + \alpha_5(t)(\sigma_Z^2 + g^2\sigma_v^2 + g^2\sigma_S^2) + \alpha_8(t)\rho_N^i\sigma\sigma_Z + \alpha_{11}(t)g(\sigma_v^2 + \sigma_{iS}\sigma_S\rho_{NS}^i)\\
\lambda\alpha_1 &= \lambda\bar{v}\\
\lambda\alpha_2 &= \alpha_2'(t) + \lambda C_1 - \gamma - \frac{c(t)^2}{b(t)(N-1)}\\
\lambda\alpha_3 &= \alpha_3'(t) - \gamma - \frac{c(t)^2 f^2}{b(t)(N-1)}\\
\lambda\alpha_4 &= \alpha_4'(t) - \frac{c(t)^2 g^2}{b(t)(N-1)}\\
\lambda\alpha_5 &= \alpha_5'(t) - \frac{c(t)^2}{b(t)(N-1)}\\
\lambda\alpha_6 &= \alpha_6'(t) + \lambda C_1 - 2\gamma - \frac{2c(t)^2 f}{b(t)(N-1)}\\
\lambda\alpha_7 &= \alpha_7'(t) + \lambda C_2 - \frac{2c(t)^2 g}{b(t)(N-1)}\\
\lambda\alpha_8 &= \alpha_8'(t) + \lambda C_3 + \frac{2c(t)^2}{b(t)(N-1)}\\
\lambda\alpha_9 &= \alpha_9'(t) + \lambda C_2 - \frac{2c(t)^2 fg}{b(t)(N-1)}\\
\lambda\alpha_{10} &= \alpha_{10}'(t) + \lambda C_3 + \frac{2c(t)^2 f}{b(t)(N-1)}\\
\lambda\alpha_{11} &= \alpha_{11}'(t) + \frac{2c(t)^2 g}{b(t)(N-1)}
\end{aligned}
$$

And the methods for optimality of demand are the same as above: For demand to be optimal, we need

$$-\Phi - \Phi_{d^i}d^i + J^d_{z^{iD}} = 0.$$

Using the equilibrium expressions for Φ, and d^i, and the form of J^d, we must have

$$\frac{a(t)+c(t)G}{b(t)} + \frac{1}{b(t)(N-1)}c(t)(z^{iI} + fz^{iD} + gS^i - G) + \alpha_1(t) + 2\alpha_3(t)z^{iD} + \alpha_6(t)z^{iI} + \alpha_9(t)S^i + \alpha_{10}(t)G = 0.$$

Matching coefficients, we get

$$\begin{aligned}
\frac{a(t)}{b(t)} + \alpha_1 &= 0 \\
\frac{c(t)f}{b(t)(N-1)} + 2\alpha_3(t) &= 0 \\
\frac{c(t)}{b(t)(N-1)} + \alpha_6(t) &= 0 \\
\frac{c(t)g}{b(t)(N-1)} + \alpha_9(t) &= 0 \\
\frac{c(t)}{b(t)} - \frac{c(t)}{b(t)(N-1)} + \alpha_{10}(t) &= 0
\end{aligned}$$

This implies $-\frac{c(t)}{b(t)(N-1)} = 2\alpha_3(t)/f = \alpha_6(t) = \alpha_9(t)/g = \frac{\alpha_{10}(t)}{N-2}$. Note, $b(t) = -\frac{fc(t)^2}{2\alpha_3(t)c(t)(N-1)}$.

The ODEs simplify to

$$\begin{aligned}
\lambda\alpha_0(t) &= \alpha_0'(t) + \alpha_2(t)\sigma^2 + \alpha_4(t)(\sigma_v^2 + \sigma_{iS}^2) + \alpha_5(t)(\sigma_Z^2 + g^2\sigma_v^2 + g^2\sigma_S^2) + \alpha_8(t)\rho_N^i\sigma\sigma_Z + \alpha_{11}(t)g(\sigma_v^2 + \sigma_{iS}\sigma_S\rho_{NS}^i) \\
\lambda\alpha_1 &= \lambda\bar{v} \\
\lambda\alpha_2(t) &= \alpha_2'(t) + \lambda C_1 - \gamma + \frac{2c(t)\alpha_3(t)}{f} \\
\lambda\alpha_3(t) &= \alpha_3'(t) - \gamma + 2c(t)f\alpha_3(t) \\
\lambda\alpha_4(t) &= \alpha_4'(t) + \frac{2c(t)\alpha_3(t)g^2}{f} \\
\lambda\alpha_5(t) &= \alpha_5'(t) + \frac{2c(t)\alpha_3(t)}{f} \\
\lambda\alpha_6(t) &= \alpha_6'(t) + \lambda C_1 - 2\gamma + 4c(t)\alpha_3(t) \\
\lambda\alpha_7(t) &= \alpha_7'(t) + \lambda C_2 + \frac{4c(t)\alpha_3(t)g}{f} \\
\lambda\alpha_8(t) &= \alpha_8'(t) + \lambda C_3 - \frac{4c(t)\alpha_3(t)}{f} \\
\lambda\alpha_9(t) &= \alpha_9'(t) + \lambda C_2 + 4c(t)\alpha_3(t)g \\
\lambda\alpha_{10}(t) &= \alpha_{10}'(t) + \lambda C_3 - 4c(t)\alpha_3(t) \\
\lambda\alpha_{11}(t) &= \alpha_{11}'(t) - \frac{4c(t)g\alpha_3(t)}{f}
\end{aligned}$$

We have $\alpha_{10}(t) = 2(N-2)\alpha_3(t)/f$ so the equation for α_{10} becomes

$$\lambda\alpha_3(t) = \alpha_3'(t) + \frac{f\lambda C_3}{2(N-2)} - \frac{2f}{N-2}\alpha_3(t)c(t).$$

Subtracting this from the equation for $\alpha_3(t)$, we get

$$0 = -\gamma - \frac{f\lambda C_3}{2(N-2)} + 2c(t)f\alpha_3 + \frac{2f}{N-2}\alpha_3 c(t) = -\gamma - \frac{f\lambda C_3}{2(N-2)} + 2\alpha_3 c(t) f\frac{N-1}{N-2}.$$

so $\alpha_3(t)c(t) = \frac{\gamma(N-2)}{2f(N-1)} + \frac{\lambda C_3}{4(N-1)}$.

Now $\alpha_9(t) = 2\alpha_3(t)g/f$ so, assuming g is constant, the equation for $\alpha_9(t)$ becomes

$$r\alpha_3(t) = \alpha_3'(t) + \frac{\lambda C_2 f}{2g} + 2\alpha_3(t)c(t)f.$$

Subtracting this from the equation for $\alpha_3(t)$, we get

$$g = -\frac{\lambda C_2 f}{2\gamma}.$$

Note g is a constant.

And, $\alpha_6(t) = \frac{2\alpha_3(t)}{f}$, so the equation for $\alpha_6(t)$ becomes

$$\lambda\alpha_3(t) = \alpha_3'(t) + \frac{f\lambda C_1}{2} - \gamma f + 2c(t)f\alpha_3(t).$$

This implies

$$\frac{f\lambda C_1}{2} - \gamma f = -\gamma,$$

so

$$f = -\frac{2\gamma}{\lambda C_1 - 2\gamma}$$

Ok, so we've solved for f and thus g, and call $h \equiv 2c\alpha_3$. Note h is a constant. Now,

$$\alpha_3(t) = A_3 e^{\lambda t} - \gamma/\lambda + hf/\lambda.$$

Solution of α_3 allows for solution to the remaining the equations. Summarizing,

$$\begin{aligned}
\alpha_1 &= \bar{v} \\
\alpha_2(t) &= A_2 e^{\lambda t} + C_1 - \gamma/\lambda + \frac{h}{f\lambda} \\
\alpha_3(t) &= A_3 e^{\lambda t} - \gamma/\lambda + hf/\lambda \\
\alpha_4(t) &= A_4 e^{\lambda t} + \frac{hg^2}{\lambda f} \\
\alpha_5(t) &= A_5 e^{\lambda t} + \frac{h}{\lambda f} \\
\alpha_6(t) &= \frac{2A_3}{f} e^{\lambda t} - \frac{2\gamma}{f\lambda} + 2h/\lambda \\
\alpha_7(t) &= A_7 e^{\lambda t} + C_2 + \frac{2hg}{\lambda f} \\
\alpha_8(t) &= A_8 e^{\lambda t} + C_3 - \frac{2h}{\lambda f} \\
\alpha_9(t) &= \frac{2A_3 g}{f} e^{\lambda t} - \frac{2\gamma g}{f\lambda} + 2hg/\lambda \\
\alpha_{10}(t) &= \frac{2A_3(N-2)}{f} e^{\lambda t} - \frac{2\gamma(N-2)}{f\lambda} + 2h(N-2)/\lambda \\
\alpha_{11}(t) &= A_{11} e^{\lambda t} - \frac{2hg}{f\lambda} \\
\alpha_0(t) &= A_0 e^{\lambda t} - Bte^{\lambda t} + C/\lambda,
\end{aligned}$$

for some constants B, C, where

$$\begin{aligned}
C = (C_1 - \gamma/\lambda + \frac{h}{f\lambda})\sigma^2 + \frac{hg^2}{f\lambda}(\sigma_v^2 + \sigma_{iS}^2) \\
+ \frac{h}{f\lambda}(\sigma_Z^2 + g^2\sigma_v^2 + g^2\sigma_S^2) \\
+ (C_3 - \frac{2h}{\lambda f})\rho_N^i \sigma\sigma_Z - \frac{2hg}{f\lambda} g(\sigma_v^2 + \sigma_{iS}\sigma_S\rho_{NS}^i).
\end{aligned}$$

and

$$B = A_2\sigma^2 + A_4(\sigma_v^2 + \sigma_{iS}^2) + A_5(\sigma_Z^2 + g^2\sigma_v^2 + g^2\sigma_S^2) + A_8\rho_N^i\sigma\sigma_Z + A_{11}g(\sigma_v^2 + \sigma_{iS}\sigma_S\rho_{NS}^i)$$

Solving the value function overnight is a bit more straightforward. We'll assume continuous signals still arrive. Let $\hat{G} = E[\bar{G}|t]$. It is straightforward to show that $\hat{G}$ evolves according to

$$d\hat{G} = \begin{pmatrix} \sigma_Z\sigma\rho_N^i & g(\sigma_v^2 + \sigma_S\sigma_{iS}\rho_{NS}^i) \end{pmatrix} \begin{pmatrix} \sigma^{-2} & 0 \\ 0 & (\sigma_v^2 + \sigma_{iS}^2)^{-1} \end{pmatrix} \begin{pmatrix} dz^{iI} \\ dS^i \end{pmatrix} = \frac{\sigma_Z\rho_N^i}{\sigma} dz^{iI} + g\frac{\sigma_v^2 + \sigma_S\sigma_{iS}\rho_{NS}^i}{\sigma_v^2 + \sigma_{iS}^2} dS^i,$$

and the conditional variance of $\hat{G}$ evolves according to

$$dv = \sigma_Z^2 + g^2(\sigma_S^2 + \sigma_v^2) - ((\sigma_Z \rho_N^i)^2 + g^2 \frac{(\sigma_v^2 + \sigma_S \sigma_{iS} \rho_{NS}^i)^2}{\sigma_v^2 + \sigma_{iS}^2})$$
$$= \sigma_Z^2(1 - (\rho_N^i)^2) + g^2(\sigma_S^2 + \sigma_v^2)\left(1 - \left(\frac{\sigma_v^2 + \sigma_S \sigma_{iS} \rho_{NS}^i}{\sigma_v^2 + \sigma_{iS}^2}\right)^2\right).$$

So,

$$v_1 = \left(\sigma_Z^2(1 - (\rho_N^i)^2) + g^2(\sigma_S^2 + \sigma_v^2)\left(1 - \left(\frac{\sigma_v^2 + \sigma_S \sigma_{iS} \rho_{NS}^i}{\sigma_v^2 + \sigma_{iS}^2}\right)^2\right)\right)\Delta.$$

Then, traders also need to solve a simple learning problem overnight. The state variable for the learning problem is

$$\theta = D$$

and the observation is

$$\xi = S^i.$$

Then,

$$d\theta = \begin{pmatrix} \sigma_v & 0 \end{pmatrix} \begin{pmatrix} dB^v \\ dB^{iS} \end{pmatrix}$$

and

$$d\xi = \begin{pmatrix} \sigma_v & \sigma_{iS} \end{pmatrix} \begin{pmatrix} dB^v \\ dB^{iS}. \end{pmatrix}$$

Rational learning implies

$$dE_t[v_t] = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2} dS^i.$$

We will also need $\bar{m} = E_{1-\Delta}[v_{1-\Delta}|1-\Delta] - \frac{\sigma_v^2}{\sigma_v^2+\sigma_{iS}^2} S_{1-\Delta}^i = \bar{v} + C_1 z_{1-\Delta}^{iD} + \left(C_2 - \frac{\sigma_v^2}{\sigma_v^2+\sigma_{iS}^2}\right) S_{1-\Delta}^i + C_3 G_{1-\Delta}$ to be a state variable. Call the value function J^n, and conjecture the following form:

$$J^n(t, z^{iI}, z^{iD}, S^i, \hat{G}, m) = \beta_0(t) + \beta_1(z^{iI} + z^{iD}) + \beta_2(t)(z^{iI})^2 + \beta_3(t)(z^{iD})^2 + \beta_4(t)(S^i)^2 + \beta_5(t)\hat{G}^2 +$$
$$+ \beta_6(t) z^{iI} z^{iD} + \beta_7(t) z^{iI} S^i + \beta_8(t) z^{iI} \hat{G} + \beta_9(t) z^{iD} S^i + \beta_{10}(t) z^{iD} \hat{G} + \beta_{11}(t) S^i \hat{G} + \beta_{12}(t)(z^{iI} + z^{iD})\bar{m}.$$

The HJB equation is

$$\lambda J^n = J_t^n + \lambda(z^{iI} + z^{iD})(m + \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2} S^i) - \gamma(z^{iI} + z^{iD})^2 + \frac{1}{2} J_{z^{iI} z^{iI}}^n \sigma^2 + \frac{1}{2} J_{S^i S^i}^n (\sigma_v^2 + \sigma_{iS}^2)$$
$$+ \frac{1}{2} J_{\hat{G}\hat{G}}^n ((\sigma_Z \rho_N^i)^2 + g^2 \frac{(\sigma_v^2 + \sigma_S \sigma_{iS} \rho_{NS}^i)^2}{\sigma_v^2 + \sigma_{iS}^2}) + J_{z^{iI}\hat{G}}^n \rho_N^i \sigma_Z \sigma + J_{S^i \hat{G}}^n g(\sigma_v^2 + \sigma_S \sigma_{iS} \rho_{NS}^i).$$

Matching coefficients, we get

$$
\begin{aligned}
\lambda\beta_0(t) &= \beta_0'(t) + \beta_2(t)\sigma^2 + \beta_4(t)(\sigma_v^2 + \sigma_{iS}^2) + \beta_5(t)((\sigma_Z\rho_N^i)^2 + g^2\frac{(\sigma_v^2 + \sigma_S\sigma_{iS}\rho_{NS}^i)^2}{\sigma_v^2 + \sigma_{iS}^2}) + \beta_8(t)\rho_N^i\sigma_Z\sigma \\
&\quad + \beta_{11}(t)g(\sigma_v^2 + \sigma_S\sigma_{iS}\rho_{NS}^i) \\
\lambda\beta_1(t) &= \lambda\bar{v} \\
\lambda\beta_2(t) &= \beta_2'(t) - \gamma \\
\lambda\beta_3(t) &= \beta_3'(t) - \gamma \\
\lambda\beta_6(t) &= \beta_6'(t) - 2\gamma \\
\lambda\beta_7(t) &= \beta_7'(t) + \lambda\frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2} \\
\lambda\beta_9(t) &= \beta_9'(t) + \lambda\frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2} \\
\lambda\beta_{12}(t) &= \beta_{12}'(t) + \lambda \\
\lambda\beta_j(t) &= \beta_j'(t) \quad j = 4, 5, 8, 10, 11.
\end{aligned}
$$

Solutions are straightforward, and as follows:

$$
\begin{aligned}
\beta_1 &= \bar{v} \\
\beta_2(t) &= -\frac{\gamma}{\lambda} + B_2e^{\lambda t} \\
\beta_3(t) &= -\frac{\gamma}{\lambda} + B_3e^{\lambda t} \\
\beta_6(t) &= -\frac{2\gamma}{\lambda} + B_6e^{\lambda t} \\
\beta_7(t) &= \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2} + B_7e^{\lambda t} \\
\beta_9(t) &= \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2} + B_9e^{\lambda t} \\
\beta_{12}(t) &= 1 + B_{12}e^{\lambda t} \\
\beta_j(t) &= B_je^{\lambda t}, \quad j = 4, 5, 8, 10, 11, \\
\beta_0(t) &= B_0e^{\lambda t} - \frac{\gamma\sigma^2}{\lambda^2} - t(B_2\sigma^2 + B_4(\sigma_v^2 + \sigma_{iS}^2) + B_5((\sigma_Z\rho_N^i)^2 \\
&\quad + g^2\frac{(\sigma_v^2 + \sigma_S\sigma_{iS}\rho_{NS}^i)^2}{\sigma_v^2 + \sigma_{iS}^2}) + B_8\rho_N^i\sigma_Z\sigma + B_{11}g(\sigma_v^2 + \sigma_S\sigma_{iS}\rho_{NS}^i))e^{\lambda t}
\end{aligned}
$$

Appendix C.1. Boundary Conditions

The value matching boundary conditions are

$$
J_{1-\Delta}^d = J_{1-\Delta}^n, \quad E[J_1^d|\mathcal{I}_{1-}] = J_{1-}^n.
$$

Let's start at $t = 1 - \Delta$. As in the main model, it's not too difficult to show that we must have $c(1-\Delta) = -\infty$, and $z^i = -fz^{iD} - gS^i + G$ at $1-\Delta$. Thus, at $1-\Delta$ the state space contracts, and we only need

$$\alpha_0(1-\Delta) = \beta_0(1-\Delta),$$

$$\alpha_3(1-\Delta)+f^2\alpha_2(1-\Delta)-f\alpha_6(1-\Delta) = \beta_3(1-\Delta)+f^2(\beta_2(1-\Delta)+\beta_{12}(1-\Delta)C_1)-f(\beta_6(1-\Delta)+\beta_{12}(1-\Delta)C_1),$$

$$\alpha_4(1-\Delta) + g^2\alpha_2(1-\Delta) - g\alpha_7(1-\Delta) = \beta_4(1-\Delta) + g^2(\beta_2(1-\Delta) + \beta_{12}(1-\Delta)C_1)$$
$$-g(\beta_7(1-\Delta) + \beta_{12}(1-\Delta)(C_2 - \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2})),$$

$$\alpha_2(1-\Delta)+\alpha_5(1-\Delta)+\alpha_8(1-\Delta) = (\beta_2(1-\Delta)+\beta_{12}(1-\Delta)C_1)+\beta_5(1-\Delta)+(\beta_8(1-\Delta)+\beta_{12}(1-\Delta)C_3),$$

$$2gf\alpha_2(1-\Delta) - g\alpha_6(1-\Delta) - f\alpha_7(1-\Delta) + \alpha_9(1-\Delta) = 2gf(\beta_2(1-\Delta) + \beta_{12}(1-\Delta)C_1)$$
$$-g(\beta_6(1-\Delta)+\beta_{12}(1-\Delta)C_1)-f(\beta_7(1-\Delta)+\beta_{12}(1-\Delta)(C_2-\frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2}))+(\beta_9(1-\Delta)+\beta_{12}(1-\Delta)(C_2-\frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2}))$$

$$-2f\alpha_2(1-\Delta) + \alpha_6(1-\Delta) - f\alpha_8(1-\Delta) + \alpha_{10}(1-\Delta) = -2f(\beta_2(1-\Delta) + \beta_{12}(1-\Delta)C_1)$$
$$+(\beta_6(1-\Delta) + \beta_{12}(1-\Delta)C_1) - f(\beta_8(1-\Delta) + \beta_{12}(1-\Delta)C_3) + (\beta_{10}(1-\Delta) + \beta_{12}(1-\Delta)C_3)$$

$$-2g\alpha_2(1-\Delta) + \alpha_7(1-\Delta) - g\alpha_8(1-\Delta) + \alpha_{11}(1-\Delta) = -2g(\beta_2(1-\Delta) + \beta_{12}(1-\Delta)C_1)$$
$$+(\beta_7(1-\Delta) + \beta_{12}(1-\Delta)(C_2 - \frac{\sigma_v^2}{\sigma_v^2 + \sigma_{iS}^2})) - g(\beta_8(1-\Delta) + \beta_{12}(1-\Delta)C_3) + \beta_{11}(1-\Delta)$$

$c(1-\Delta) = -\infty$ is equivalent to $\alpha_3(1-\Delta) = 0$, so this pins down A_3.

The boundary conditions are simpler, and require $\beta_i(1) = \alpha_i(0)$ for $i = 1, \ldots, 11$. And,

$$\beta_0(1) = \alpha_0(0) + \alpha_5(0)v_1,$$

where v_1 is given above. We also have $\beta_{12}(1) = 0$.

The solutions are unique, and except for A_0 and B_0, are reported below.

$$A_2 = \frac{e^{-\lambda}\left(e^{\Delta\lambda}\left(e^{\lambda}((2f-1)\gamma - f(C_1\lambda+h)) + fh - \gamma\right) + e^{\lambda}(C_1 f\lambda - 2(f-1)\gamma)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$A_3 = \frac{e^{-\lambda(1-\Delta)}(\gamma - fh)}{\lambda},$$

$$A_4 = \frac{e^{-\lambda}\left(ge^{\lambda}(C_2 f\lambda + 2g\gamma) - ge^{\Delta\lambda}\left(C_2 fe^{\lambda}\lambda + g\left(e^{\lambda}(fh+\gamma) - fh + \gamma\right)\right)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$A_5 = \frac{e^{-\lambda}\left(e^{\Delta\lambda}\left((2N-3)(\gamma - fh) - e^{\lambda}(f(h - C_3\lambda) + \gamma)\right) - e^{\lambda}(C_3 f\lambda - 2fh(N-1) + 2\gamma(N-2))\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$A_7 = \frac{e^{-\lambda}\left(e^{\Delta\lambda}\left(e^{\lambda}(-f\lambda(C_1 g + C_2) + 2(f-1)g\gamma - 2fgh) - 2g(\gamma - fh)\right) + e^{\lambda}(f\lambda(C_1 g + C_2) - 2(f-2)g\gamma)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$A_8 = \frac{e^{-\lambda}\left(e^{\Delta\lambda}\left(e^{\lambda}(f\lambda(C_1 - C_3) - 2(f-1)\gamma + 2fh) - 2(N-2)(\gamma - fh)\right) + e^{\lambda}(f\lambda(C_3 - C_1) + 2\gamma(f+N-3) - 2fh(N-1))\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$A_{11} = \frac{e^{-\lambda}\left(e^{\Delta\lambda}\left(e^{\lambda}(f\lambda(C_2 - C_3 g) + 2g(fh+\gamma)) - 2g(N-2)(\gamma - fh)\right) + e^{\lambda}(-C_2 f\lambda + fg(C_3\lambda - 2h(N-1)) + 2g\gamma(N-3))\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_2 = \frac{e^{-2\lambda}\left(e^{\Delta\lambda}\left(e^{\lambda}((2f-1)\gamma - f(C_1\lambda+h)) + fh - \gamma\right) + e^{\lambda}\left(C_1 f\lambda\left(f\left(e^{\lambda}-1\right)+1\right) - 2(f-1)\gamma + fh\left(e^{\lambda}-1\right)\right)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_3 = \frac{e^{-\lambda(2-\Delta)}(\gamma + (e^{\lambda(1-\Delta)} - 1)fh)}{\lambda},$$

$$B_4 = \frac{ge^{-2\lambda}\left(e^{\lambda}\left(C_2 f\lambda + fgh\left(e^{\lambda}-1\right) + 2g\gamma\right) - e^{\Delta\lambda}\left(C_2 fe^{\lambda}\lambda + g\left(e^{\lambda}(fh+\gamma) - fh + \gamma\right)\right)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_5 = \frac{e^{-\lambda}\left(C_3 f\lambda\left(e^{\Delta\lambda}-1\right) - fh\left(e^{(\Delta-1)\lambda}-1\right)\left(e^{\lambda}+2N-3\right) - \gamma e^{(\Delta-1)\lambda}\left(e^{\lambda}-2N+3\right) - 2\gamma(N-2)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_6 = \frac{2e^{-2\lambda}\left(e^{\Delta\lambda}(\gamma - fh) + e^{\lambda}((f-1)\gamma + fh)\right)}{f\lambda},$$

$$B_7 = \frac{e^{-2\lambda}\Big(e^{\Delta\lambda}\left(e^{\lambda}(-f\lambda(C_1 g + C_2) + 2(f-1)g\gamma - 2fgh) - 2g(\gamma - fh)\right)}{f^2\left(e^{\lambda}-1\right)\lambda}$$
$$+ \frac{e^{\lambda}\left(g\left(C_1 f\lambda - 2(f-2)\gamma + 2fh\left(e^{\lambda}-1\right)\right) + f\lambda\left(fe^{\lambda}(C_2 - \frac{\sigma_v^2}{\sigma_v^2+\sigma_{iS}^2}) + C_2(-f) + C_2 + f\frac{\sigma_v^2}{\sigma_v^2+\sigma_{iS}^2}\right)\right)\Big)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_8 = \frac{e^{-\lambda}\Big(f\left(\lambda\left((C_1 - C_3)e^{\Delta\lambda} - C_1 + C_3 fe^{\lambda} - C_3 f + C_3\right) + 2h\left(e^{(\Delta-1)\lambda}-1\right)\left(e^{\lambda}+N-2\right)\right)}{f^2\left(e^{\lambda}-1\right)\lambda}$$
$$+ \frac{2\gamma\left((f-1)\left(-e^{\Delta\lambda}\right) - (N-2)e^{(\Delta-1)\lambda} + f + N - 3\right)\Big)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_9 = \frac{e^{-2\lambda}\left(2ge^{\Delta\lambda}(\gamma - fh) - e^{\lambda}(2g(\gamma - fh) + f\lambda\frac{\sigma_v^2}{\sigma_v^2+\sigma_{iS}^2})\right)}{f\lambda},$$

$$B_{10} = \frac{2e^{-2\lambda}(N-2)\left(e^{\Delta\lambda} - e^{\lambda}\right)(\gamma - fh)}{f\lambda},$$

$$B_{11} = \frac{e^{-\lambda}\left(C_2 f\lambda\left(e^{\Delta\lambda}-1\right) + fg\left(2h\left(e^{(\Delta-1)\lambda}-1\right)\left(e^{\lambda}+N-2\right) - C_3\lambda\left(e^{\Delta\lambda}-1\right)\right) + 2g\gamma\left(e^{(\Delta-1)\lambda}\left(e^{\lambda}-N+2\right) + N - 3\right)\right)}{f^2\left(e^{\lambda}-1\right)\lambda},$$

$$B_{12} = -e^{-\lambda}.$$